Hidden symmetries and gauge structure of Yang-Mills theories with compactified extra dimensions

M. A. López-Osorio[(a)], E. Martínez-Pascual[(a)], H. Novales-Sánchez[(b)], and J. J. Toscano[(a)]
[(a)] *Facultad de Ciencias Físico Matemáticas, Benemérita Universidad Autónoma de Puebla, Apartado Postal 1152, Puebla, Puebla, México.*
[(b)] *Facultad de Ciencias Físico Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Avenida Francisco J. Mújica S/N, 58060, Morelia, Michoacán, México.*

In a previous paper by some of us [H. Novales-Sánchez and J. J. Toscano, Phys. Rev. D **82**, 116012 (2010)], the gauge structure of Yang-Mills theories with one universal extra dimension was explored. In particular, two types of gauge invariance were identified and classified as standard gauge transformations (SGT) and nonstandard gauge transformations (NSGT). The main purpose of this work is to give a precise meaning to this classification within the context of hidden symmetries. In three different gauge systems, suitable canonical transformations capable of hiding explicit symmetries are found. The systems under consideration are: (i) four dimensional pure $SU(3)$ Yang-Mills theory, (ii) four dimensional $SU(3)$ Yang-Mills with spontaneous symmetry breaking, and (iii) pure Yang-Mills theory with one universal compact extra dimension. In all cases the original system is mapped into a certain effective theory that is invariant under the so–called SGT and NSGT. In the case where spontaneous symmetry breaking is present, the set of SGT corresponds to the group to which the original gauge group is broken into, whereas the NSGT are associated to the broken generators. System (ii) is a particular case of the more general scenario in which the symmetry G is broken in to H and a canonical transformation is introduced to map covariant objects under G into covariant objects under H. For systems (i) and (ii) the group of SGT is $SU(2)$, whereas for the system (iii) this group is referred to as $SU(N, \mathcal{M}^4)$, the standard $SU(N)$ with gauge parameters defined on four dimensional Minkowski spacetime. Prospects to generalize the system (iii) to more than one extra dimension are also discussed. The differences between the pseudo Goldstone bosons that emerge from a degenerate vacuum and those induced by compactification are stressed.

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I. INTRODUCTION

Recently, the ATLAS [1] and CMS [2] experiments at the Large Hadron Collider (LHC) reported the presence of a scalar boson, with mass in the range $125-126\,GeV$, that is compatible with the Standard Model's (SM) Higgs boson. If couplings of this particle to pairs of W and Z weak gauge bosons are found to coincide with those predicted by the SM in subsequent analysis of experimental data, the Higgs mechanism [3] will be firmly established as a genuine phenomenon of nature. Since Higgs mechanism endows gauge bosons in a gauge theory with mass through absorption of Goldstone bosons [4], arising in spontaneous symmetry breaking (SSB), it would confirm that the weak interaction is spontaneously broken. This in turn would validate the existence of a degenerate vacuum as the source of elementary particle masses. This constitutes a good motivation to investigate new mechanisms of mass generation. In particular, it is interesting to study some kind of source, alternative to spontaneous symmetry breaking, that however allows the Higgs mechanism to operate. On this direction, it is already known that gauge theories formulated on spacetime manifolds with compact extra dimensions [5] allow us to endow with mass the Kaluza-Klein gauge excitations in the absence of degenerate vacuum. Although in these theories there are pseudo–Goldstone bosons, allowing thus the Higgs mechanism to operate, they do not correspond to genuine Goldstone bosons in the sense of spontaneous breakdown of a global symmetry. The emergent Goldstone bosons in Kaluza-Klein theories are directly generated by compactification of the extra–spatial dimensions.

Our main goal in this work is to clarify the gauge structure of pure Yang-Mills theories formulated on flat spacetime manifolds with compact spatial extra dimensions. Some theoretical aspects of these theories have already been studied in Refs. [6–9]. Also, they have been the subject of important phenomenological interest in the contexts of dark matter [10], neutrino Physics [11], Higgs physics [12], flavor physics [13], Hadronic and linear colliders [14], and electroweak gauge couplings [15]. In Ref. [6], some results in the context of a pure Yang-Mills theory with one universal extra dimension (UED) were presented; it was emphasized the necessity of explaining the gauge structure of the compactified theory in order to quantize it. In particular, it was obtained that as a consequence of compactification, the original gauge transformations split into two classes of gauge transformations: the standard gauge transformations (SGT) and the nonstandard gauge transformations (NSGT). In the present work we will show that a notion of hidden symmetry can be merged into the canonical structure not only of compactified theories, but also of gauge theories where no compactification scheme is given.

The concept of hidden symmetry is usually associated to theories in which SSB occurs; however, we will show that this is not exclusive of this kind of theories. A symmetry, encoded in a gauge group G, can also be hidden if there is a canonical transformation that maps well defined objects under the group G to well defined objects under a subgroup H. As we will see below, the possibility of mapping covariant objects of G into covariant objects of its subgroup H is crucial to understand a hidden symmetry in this context. It is at the level of the remaining symmetry that SGT and NSGT find a clear interpretation. The set of SGT forms a group which coincides with the subgroup H of G, whereas the set of NSGT does not form a group and is associated with those generators of G which do not belong to H. The phenomenon of SSB can be fit into this general scenario when a scalar sector that leads to a degenerate vacuum which is invariant under the subgroup H is introduced. In this case, the NSGT are associated to the broken generators of the group G.

To clarify the ideas developed in the previous paragraph, we study in detail three gauge models. In order to show that our notion of hidden symmetry is not necessarily a consequence of a compactification scheme from a higher dimensional theory, the four dimensional pure $SU(3)$ Yang-Mills theory is considered. The clarity in the analysis of this model permits us to use it as a toy model, where $G = SU(3)$ and $H = SU(2)$. In the sense defined above, a suitable canonical transformation is constructed; in particular, gauge fields with respect to $SU(3)$ (A^a_μ, $a = 1, \ldots, 8$) are transformed into gauge fields ($W^{\bar{a}}_\mu$), two doublets (Y_μ and $Y^\dagger_\mu$) and a singlet (Z_μ) all with respect to $SU(2)$. The $SU(3)$ symmetry is hidden into the SGT and NSGT which correspond to transformations in $SU(2)$ and transformations related with the five remainder generators of $SU(3)$, respectively. This analysis, which does not involve SSB, will play a central role when we study Yang-Mills theories with compactified extra dimensions.

In order to clarify the physical meaning of the NSGT, the study of the $SU(3)$ Yang-Mills theory with a renormalizable scalar sector that presents SSB is achieved. Besides the decomposition of the $SU(3)$ Yang-Mills connection and the matter scalar into well defined objects with specific transformation rules under the subgroup $SU(2)$, in this model it is shown that the unitary gauge corresponds to a particular NSGT which maps the pseudo Goldstone bosons into zero. Although, in gauge theories with compactified extra dimensions the corresponding NSGT are not associated to broken generators, we will see that the pseudo Goldstone bosons that arise in this class of theories can be also mapped into zero through NSGT, allowing thus that the Higgs mechanism operates.

The third system in which we focus our attention is a pure Yang-Mills theory with various compact extra dimensions. We assume the basic gauge fields $\mathcal{A}^a_M$, as well as gauge group parameters, to be defined on an m dimensional manifold $\mathcal{M} = \mathcal{M}^4 \times \mathcal{N}^n$, with n the number of extra dimensions. Throughout this paper we work with a metric signature that is mostly negative, that is, $\mathrm{diag}(1, -1, \ldots, -1)$. We denote the gauge group of these theories as $SU(N, \mathcal{M})$ to emphasize that all gauge parameters of the Lie group $SU(N)$ are defined on $\mathcal{M}$. The effective theory that results from integrating out the extra dimensions is invariant under SGT and NSGT, the set of the former transformations constitutes $SU(N)$ but now defined on $\mathcal{M}^4$, we denote this group by $SU(N, \mathcal{M}^4)$. The restriction of gauge parameters of $SU(N, \mathcal{M})$ to the submanifold $\mathcal{M}^4$ defines the subgroup $SU(N, \mathcal{M}^4)$. The effective theory is characterized by the zero–mode gauge fields $A^{(0,\ldots,0)a}_\mu(x)$ and the infinite tower of Kaluza-Klein (KK) excitations $A^{(m,\ldots)a}_\mu(x)$ and $A^{(m,\ldots)a}_{\bar{\mu}}(x)$, where the whole set of spacetime indices in a local chart of $\mathcal{M}$ is split into those of Minkowski spacetime μ and the extra dimensions $\bar{\mu}$. The $\mathcal{A}^a_M$ gauge fields (and their canonical conjugate momenta) are related to the basic fields in the effective theory via a Fourier series. We show that this relation is a canonical transformation such that each mode in the expansion is an object with a well determined transformation rule under $SU(N, \mathcal{M}^4)$, making this transformation in phase space a suitable one in the sense explained above. This canonical transformation permeates even at the level of the Dirac algorithm: it is shown that each generation of constraints as well as the primary Hamiltonian of the pure $SU(N, \mathcal{M})$ Yang-Mills theory is mapped into the respective generation of constraints and primary Hamiltonian unfolded in the Dirac analysis of the $SU(N, \mathcal{M}^4)$ based theory. This fact implies that the theories before and after compactification are physically equivalent, this is one of the main result in the present paper. The gauge structure of the higher–dimensional theory is just hidden in the lower–dimensional theory, that is, $SU(N, \mathcal{M})$ is certainly codified into the SGT and NSGT. It is worth noticing that the transition from the $SU(N, \mathcal{M})$ description to the $SU(N, \mathcal{M}^4)$ one does not involve SSB because the number of generators of the groups matches. Compactification only implies a change in the support manifold of the group parameters.

The rest of the paper has been organized as follows. In Sec. II, the pure $SU(3)$ Yang-Mills theory is introduced, the canonical analysis of the theory before and after considering a suitable canonical transformation is independently achieved. It is shown that both frameworks lead to the same theory with the same number of physical degrees of freedom and the same gauge transformations,*i.e.* the canonical transformation simply recasts the system. In Sec. III a renormalizable scalar Higgs sector is added to the model presented in Sec. II, the corresponding suitable canonical transformation is introduced. We show that the presence of spontaneous breakdown $SU(3) \to SU(2)$ allows us to use a specific NSGT to fix the unitary gauge. Section IV is devoted to the study of pure $SU(N, \mathcal{M})$ Yang-Mills theory; with the more tractable case of one UED, we explicitly present the suitable canonical transformation and compactification scheme that led us to the effective theory invariant under SGT and NSGT, we argue that both theories are equivalent as they have the same gauge transformations, simply written in different coordinates, and contain the same number

of physical degrees of freedom. In Sec. V a summary of our results is presented. Finally, in Appendix A we collect the proof on the canonical nature of the Fourier transform.

II. THE TOY MODEL: PURE $SU(3)$ YANG-MILLS THEORY

The purpose of this section is to illustrate the notion of hidden symmetry within the context described in the Introduction, for which we consider the situation where $G = SU(3)$ and $H = SU(2)$. This model has attracted important phenomenological interest within the context of the so-called 331 models [16] and it is interesting for us because the $SU(2)$ group is completely embedded in the $SU(3)$ one. This feature allows us to illustrate more clearly all the peculiarities of a hidden symmetry.

A. The $SU(3)$ perspective of the model

We consider the four–dimensional Yang-Mills theory based on the group $SU(3)$ with the well-known Lagrangian

$$\mathcal{L}_{SU(3)} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a \ , \tag{II.1}$$

where the components of the Yang-Mills curvature are given in terms of the gauge fields A^a_μ by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \ . \tag{II.2}$$

In the special case of $SU(3)$, the completely antisymmetric structure constants f^{abc} have the following nonvanishing values: $f^{123} = 1$, $f^{147} = -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = \frac{1}{2}$ and $f^{458} = f^{678} = \frac{\sqrt{3}}{2}$.

The Lagrangian (II.1) is invariant under the gauge transformations

$$\delta A^a_\mu(x) = \mathcal{D}^{ab}_\mu \alpha^b(x) \ , \tag{II.3}$$

where $\alpha^a(x)$ are the gauge parameters of the group and $\mathcal{D}^{ab}_\mu = \delta^{ab}\partial_\mu - g f^{abc} A^c_\mu$ is the covariant derivative in the adjoint representation. The above gauge transformations imply that the components of the curvature transform in the adjoint representation of the group,

$$\delta F^a_{\mu\nu} = g f^{abc} F^b_{\mu\nu} \alpha^c \ . \tag{II.4}$$

As far as the Hamiltonian structure of the theory is concerned, the canonical momenta are defined by

$$\pi^\mu_a \equiv \frac{\partial \mathcal{L}_{SU(3)}}{\partial \dot{A}^a_\mu} = F^{\mu 0}_a \ , \tag{II.5}$$

where the dot over the fields denotes time derivative. This expression immediately leads to the following primary constraints:

$$\phi^{(1)}_a \equiv \pi^0_a \approx 0 \ . \tag{II.6}$$

The time evolution along the motion of an arbitrary function on the phase space is dictated by the primary Hamiltonian

$$H^{(1)} = \int d^3x \, \mathcal{H}^{(1)}_{SU(3)} \ , \tag{II.7}$$

where

$$\mathcal{H}^{(1)}_{SU(3)} = \mathcal{H}_{SU(3)} + \mu^a \phi^{(1)}_a \ , \tag{II.8}$$

μ^a are Lagrange multipliers, and $\mathcal{H}_{SU(3)}$ is the canonical Hamiltonian. The latter is

$$\mathcal{H}_{SU(3)} = \frac{1}{2}\pi^i_a \pi^i_a + \frac{1}{4} F^a_{ij} F^{ij}_a - A^a_0 \phi^{(2)}_a \ . \tag{II.9}$$

Any physically allowed initial configuration of fields and conjugate momenta must satisfy the primary constraints (II.6), hence the constraints must be constant in time. This consistency condition on the primary constraints leads to the following secondary constraints:

$$\phi_a^{(2)} \equiv \mathcal{D}_i^{ab}\pi_b^i \approx 0\,. \tag{II.10}$$

Applying the consistency condition to secondary constraints yields no new constraints. In this case, all the constraints are of first–class type [17]; the Poisson brackets among the constraints are linear combinations of the constraints themselves. The nonvanishing Poisson brackets between the first–class constraints are

$$\{\phi_a^{(2)}[u], \phi_b^{(2)}[v]\}_{SU(3)} = g f_{abc}\,\phi_c^{(2)}[uv]\ , \tag{II.11}$$

where smeared form of the constraints was used, *e.g.* $\phi_a^{(2)}[u] := \int d^3x\, u(\mathbf{x})\phi_a^{(2)}(\mathbf{x})$. The label $SU(3)$ on the Poisson bracket indicates that it is calculated with respect to the canonical conjugate pairs (A_μ^a, π_a^μ).

As it is well known [18] the number of true degrees of freedom, in a theory with first–class constraints only, corresponds to the total number of canonical variables minus twice the number of first–class constraints, all divided by two. Therefore, the number of true degrees of freedom is in this case $(8\times 4) - (8\times 2) = 16$ per spatial point $(\mathbf{x})$.

In this system all first–class constraints generate gauge transformations (II.3) through the gauge generator [19]

$$G = (\mathcal{D}_0^{ab}\alpha^b)\phi_a^{(1)} - \alpha^a\phi_a^{(2)} \tag{II.12}$$

via the Poisson bracket as follows

$$\delta A_\mu^a = \{A_\mu^a, G\}_{SU(3)}\ . \tag{II.13}$$

We now turn to formulate the same theory but from the perspective of $SU(2)$.

B. The $SU(2)$ perspective of the model

The fundamental representation of $SU(3)$ has dimension 3. A particular choice of this representation is given by the well known Gell-mann matrices λ^a, being the corresponding generators $\lambda^a/2$. Since matrices λ^3 and λ^8 commute with each other, there are three independent $SU(2)$ subgroups, whose generators are $(\lambda^1,\lambda^2,\lambda)$, $(\lambda^4,\lambda^5,\lambda)$ and $(\lambda^6,\lambda^7,\lambda)$. For each case, λ is a different linear combination (with real coefficients) of λ^3 and λ^8. In this work, we will consider the subgroup determined by the set of generators $(\lambda^1,\lambda^2,\lambda^3)$ and the corresponding values of the structure constants will be denoted by $f^{\bar a\bar b\bar c} = \epsilon^{\bar a\bar b\bar c}$, where $\bar a = 1,2,3$. We will also use the notation $\hat a = 4,5,6,7$ so that $a = 1,\ldots,8 = \bar a, \hat a, 8$. From the $SU(2)$ perspective $\bar a$ will label gauge fields, whereas $\hat a$ and 8 will label tensorial representations of $SU(2)$, see Eqs. (II.19).

In the configuration space, we consider the following point transformation:

$$A_\mu^{\bar a} = W_\mu^{\bar a}\ , \tag{II.14a}$$

$$A_\mu^4 = \frac{1}{\sqrt{2}}\left(Y_{\mu 1}^* + Y_\mu^1\right)\ ,\ A_\mu^5 = \frac{1}{\sqrt{2}}\left(Y_{\mu 1}^* - Y_\mu^1\right)\ , \tag{II.14b}$$

$$A_\mu^6 = \frac{1}{\sqrt{2}}\left(Y_{\mu 2}^* + Y_\mu^2\right)\ ,\ A_\mu^7 = \frac{1}{\sqrt{2}}\left(Y_{\mu 2}^* - Y_\mu^2\right)\ , \tag{II.14c}$$

$$A_\mu^8 = Z_\mu\ . \tag{II.14d}$$

This mapping relates the coordinates of the $SU(3)$ formulation to the coordinates we will use in the $SU(2)$ perspective. The inverse is conveniently arranged as follows:

$$W_\mu^{\bar a} = A_\mu^{\bar a}\ , \tag{II.15a}$$

$$Y_\mu = \begin{pmatrix} Y_\mu^1 \\ Y_\mu^2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} A_\mu^4 - iA_\mu^5 \\ A_\mu^6 - iA_\mu^7 \end{pmatrix}\ , \tag{II.15b}$$

$$Y_\mu^\dagger = (Y_{\mu\,1}^*\ \ Y_{\mu\,2}^*) = \frac{1}{\sqrt{2}}(A_\mu^4 + iA_\mu^5\ \ \ A_\mu^6 + iA_\mu^7)\ , \tag{II.15c}$$

$$Z_\mu = A_\mu^8\ . \tag{II.15d}$$

As it will be confirmed below, see Eq.(II.19), fields Y_μ and $Y_\mu^\dagger$ transform as contravariant and covariant $SU(2)$ objects, respectively, whereas Z_μ becomes invariant under this group.

Using the above point transformation, the Yang-Mills curvature components (II.2) can be rearranged as follows:

$$F^{\bar{a}}_{\mu\nu} = W^{\bar{a}}_{\mu\nu} + ig\left(Y^\dagger_\mu \frac{\sigma^{\bar{a}}}{2} Y_\nu - Y^\dagger_\nu \frac{\sigma^{\bar{a}}}{2} Y_\mu\right) \ , \tag{II.16a}$$

$$Y_{\mu\nu} = D_\mu Y_\nu - D_\nu Y_\mu + ig\frac{\sqrt{3}}{2}\left(Y_\mu Z_\nu - Y_\nu Z_\mu\right) \ , \tag{II.16b}$$

$$F^{8}_{\mu\nu} = Z_{\mu\nu} + ig\frac{\sqrt{3}}{2}\left(Y^\dagger_\mu Y_\nu - Y^\dagger_\nu Y_\mu\right) \ . \tag{II.16c}$$

In these equations, $W^{\bar{a}}_{\mu\nu} = \partial_\mu W^{\bar{a}}_\nu - \partial_\nu W^{\bar{a}}_\mu + g\epsilon^{\bar{a}\bar{b}\bar{c}} W^{\bar{b}}_\mu W^{\bar{c}}_\nu$ are the components of the $su(2)$-valued curvature, $D_\mu = \partial_\mu - ig\frac{\sigma^{\bar{a}}}{2} W^{\bar{a}}_\mu$ is the covariant derivative in the fundamental representation of $SU(2)$, and $Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu$. The components $F^{\hat{a}}_{\mu\nu}$ are encoded into $Y_{\mu\nu}$.

The Lagrangian (II.1) from the $SU(2)$ perspective takes the form

$$\mathcal{L}_{SU(2)} = -\frac{1}{4} F^{\bar{a}}_{\mu\nu} F^{\mu\nu}_{\bar{a}} - \frac{1}{2} Y^\dagger_{\mu\nu} Y^{\mu\nu} - \frac{1}{4} F^{8}_{\mu\nu} F^{\mu\nu}_{8} \ . \tag{II.17}$$

In terms of covariant objects of $SU(2)$, the gauge transformations (II.3) are mapped into

$$\delta W^{\bar{a}}_\mu = \mathcal{D}^{\bar{a}\bar{b}}_\mu \alpha^{\bar{b}} - ig\left(\beta^\dagger \frac{\sigma^{\bar{a}}}{2} Y_\mu - Y^\dagger_\mu \frac{\sigma^{\bar{a}}}{2} \beta\right) \ , \tag{II.18a}$$

$$\delta Y_\mu = ig\frac{\sigma^{\bar{a}}}{2}\alpha^{\bar{a}}\, Y_\mu + \left(D_\mu - ig\frac{\sqrt{3}}{2} Z_\mu\right)\beta + ig\frac{\sqrt{3}}{2}\, Y_\mu \alpha_Z \ , \tag{II.18b}$$

$$\delta Z_\mu = \partial_\mu \alpha_Z - ig\frac{\sqrt{3}}{2}\left(\beta^\dagger Y_\mu - Y^\dagger_\mu \beta\right) \ . \tag{II.18c}$$

where $\beta^\dagger = \left(\frac{1}{\sqrt{2}}\left(\alpha^4 + i\alpha^5\right) \quad \frac{1}{\sqrt{2}}\left(\alpha^6 + i\alpha^7\right)\right)$. From the $SU(2)$ perspective, the eight parameters of $SU(3)$ split into the standard three gauge parameters, $\alpha^{\bar{a}}$, two doublets, β and $\beta^\dagger$, and a singlet, α_Z, of $SU(2)$. In Eq. (II.18a) the covariant derivative of $SU(2)$, $\mathcal{D}^{\bar{a}\bar{b}}_\mu = \delta^{\bar{a}\bar{b}}\partial_\mu - g\epsilon^{\bar{a}\bar{b}\bar{c}} W^{\bar{c}}_\mu$, in its adjoint representation emerges.

The *standard gauge transformations* (SGT) are defined from the transformation laws (II.18) by setting the parameters β and α_Z equal to zero,

$$\delta_{\rm s} W^{\bar{a}}_\mu \equiv \mathcal{D}^{\bar{a}\bar{b}}_\mu \alpha^{\bar{b}} \ , \tag{II.19a}$$

$$\delta_{\rm s} Y_\mu \equiv ig\frac{\sigma^{\bar{a}}}{2}\alpha^{\bar{a}}\, Y_\mu \ , \tag{II.19b}$$

$$\delta_{\rm s} Z_\mu \equiv 0 \ . \tag{II.19c}$$

From these expressions, it is shown $W^{\bar{a}}_\mu$ transform as gauge fields, Y_μ as a doublet of $SU(2)$, and Z_μ as an invariant under the $SU(2)$ group. This means that transformation (II.14) constitutes an admissible point transformation as covariant objects of $SU(3)$ are mapped into covariant objects of $SU(2)$. Moreover, Eqs. (II.19b) and (II.19c) make manifest that Y_μ and Z_μ are matter fields. In the context of this description, there arise *nonstandard gauge transformations* (NSGT), which are defined from (II.18) by setting $\alpha^{\bar{a}} = 0$,

$$\delta_{\rm ns} W^{\bar{a}}_\mu = -ig\left(\beta^\dagger \frac{\sigma^{\bar{a}}}{2} Y_\mu - Y^\dagger_\mu \frac{\sigma^{\bar{a}}}{2} \beta\right) \ , \tag{II.20a}$$

$$\delta_{\rm ns} Y_\mu = \left(D_\mu - ig\frac{\sqrt{3}}{2} Z_\mu\right)\beta + ig\frac{\sqrt{3}}{2}\, Y_\mu \alpha_Z \ , \tag{II.20b}$$

$$\delta_{\rm ns} Z_\mu = \partial_\mu \alpha_Z - ig\frac{\sqrt{3}}{2}\left(\beta^\dagger Y_\mu - Y^\dagger_\mu \beta\right) \ , \tag{II.20c}$$

These NSGT tell us that there is a gauge symmetry larger than $SU(2)$, in our case $SU(3)$. *More precisely, the difference between SGT and NSGT is that the former are associated with generators that constitute a group, whereas the latter have to do with generators that do not form a subgroup.* The recognizing of this fact is crucial to quantize

the theory, as it requires to incorporate the gauge parameters as degrees of freedom from the beginning, so the use of only the $SU(2)$ parameters or the complete set of the $SU(3)$ parameters would lead to very different quantized theories. Of course, the theory can be quantized using the $SU(2)$ basis but taking into account that Y_μ and Z_μ are also gauge fields, which means that the β and α_Z parameters must be recognized as (spurious) degrees of freedom in the context of the BRST [20, 21] symmetry.

In order to justify that (II.18) are actual gauge transformations, we wish to show the invariance of the Lagrangian (II.17) under these variations. Therefore one may want to start exploring the behaviour of (II.16) under (II.18). After some algebra one finds

$$\delta F^{\bar{a}}_{\mu\nu} = g\epsilon^{\bar{a}\bar{b}\bar{c}}F^{\bar{b}}_{\mu\nu}\alpha^{\bar{c}} + ig\left(Y^\dagger_{\mu\nu}\frac{\sigma^{\bar{a}}}{2}\beta - \beta^\dagger\frac{\sigma^{\bar{a}}}{2}Y_{\mu\nu}\right) , \tag{II.21a}$$

$$\delta Y_{\mu\nu} = ig\frac{\sigma^{\bar{a}}}{2}Y_{\mu\nu}\alpha^{\bar{a}} - igF^{\bar{a}}_{\mu\nu}\frac{\sigma^{\bar{a}}}{2}\beta + ig\frac{\sqrt{3}}{2}\left(Y_{\mu\nu}\alpha_Z - F^8_{\mu\nu}\beta\right) , \tag{II.21b}$$

$$\delta F^8_{\mu\nu} = ig\frac{\sqrt{3}}{2}\left(Y^\dagger_{\mu\nu}\beta - \beta^\dagger Y_{\mu\nu}\right) . \tag{II.21c}$$

It can be shown that the Lagrangian in the $SU(2)$ description (II.17) is invariant under these transformation. Therefore it is also invariant under the transformations (II.18) that decompose into the sum of SGT and NSGT.

We now place some technical comments. In the variation of $Y_{\mu\nu}$ (II.21b) the following extra term is explicitly found

$$\begin{aligned} B_{\mu\nu} =& -g^2\Big[\big(Y^\dagger_\mu\frac{\sigma^{\bar{a}}}{2}Y_\nu - Y^\dagger_\nu\frac{\sigma^{\bar{a}}}{2}Y_\mu\big)\frac{\sigma^{\bar{a}}}{2}\beta + \frac{3}{4}\big(Y^\dagger_\mu Y_\nu - Y^\dagger_\nu Y_\mu\big) + \frac{3}{4}\big(\beta^\dagger Y_\mu - Y^\dagger_\mu\beta\big)Y_\nu\beta \\ &- \frac{3}{4}\big(\beta^\dagger Y_\nu - Y^\dagger_\nu\beta\big)Y_\mu + \big(\beta^\dagger\frac{\sigma^{\bar{a}}}{2}Y_\mu - Y^\dagger_\mu\frac{\sigma^{\bar{a}}}{2}\beta\big)\frac{\sigma^{\bar{a}}}{2}Y_\nu - \big(\beta^\dagger\frac{\sigma^{\bar{a}}}{2}Y_\nu - Y^\dagger_\nu\frac{\sigma^{\bar{a}}}{2}\beta\big)\frac{\sigma^{\bar{a}}}{2}Y_\mu\Big]. \end{aligned}$$

It at first sight seems to be different from zero, but consistency between the $SU(2)$ and the $SU(3)$ perspectives of the same theory indicates that it must vanish. Indeed, using the point transformation (II.15), variations (II.21) are mapped into (II.4) as required. Also since $B_{\mu\nu}$ is linear in β, its occurrence in the variation of $Y_{\mu\nu}$ would spoil the invariance of the Lagrangian (II.17) under the NSGT (II.20); however, one can see that the rth $SU(2)$ component ($r = 1, 2$) of the doublet $B_{\mu\nu}$ is of the form

$$B^r_{\mu\nu} = -\frac{1}{4}\Big[\big(T^{rs}_{pq} - T^{rs}_{qp}\big)\big(Y^*_{\mu s}Y_\nu{}^q - Y^*_{\nu s}Y_\mu{}^q\big)\beta^p - T^{rs}_{pq}\big(Y_\mu{}^pY_\nu{}^q - Y_\nu{}^pY_\mu{}^q\big)\beta_s\Big] ;$$

where $T^{rs}_{pq} \equiv (\sigma^{\bar{a}})^r_p(\sigma^{\bar{a}})^s_q + 3\delta^r_p\delta^s_q$. Using the explicit values of the indices shows that T^{rs}_{pq} is symmetric in p and q, hence $B^r_{\mu\nu} \equiv 0$. A similar behaviour is present in the invariance of the effective Yang-Mills Lagrangian in four spacetime dimensions obtained by integrating out a compact space fifth dimensions of a pure five dimensional Yang-Mills theory described in Sec. IV. Finally, since the variations (II.21) are obtained from (II.18), at the curvature level, the SGT and NSGT are induced at the level of the curvature; in particular the SGT of $F^{\bar{a}}_{\mu\nu}$, $Y_{\mu\nu}$, and $F^8_{\mu\nu}$

$$\delta_{\rm s}F^{\bar{a}}_{\mu\nu} = g\epsilon^{\bar{a}\bar{b}\bar{c}}F^{\bar{b}}_{\mu\nu}\alpha^{\bar{c}} , \tag{II.22a}$$

$$\delta_{\rm s}Y_{\mu\nu} = ig\frac{\sigma^{\bar{a}}}{2}Y_{\mu\nu}\alpha^{\bar{a}} , \tag{II.22b}$$

$$\delta_{\rm s}F^8_{\mu\nu} = 0 , \tag{II.22c}$$

imply the previously enunciated fact: $F^{\bar{a}}_{\mu\nu}$, $Y_{\mu\nu}$, and $F^8_{\mu\nu}$ transform in the adjoint, fundamental, and trivial representation of $SU(2)$, respectively.

It is interesting to note that one can fix the gauge for the Y_μ fields in a covariant way under the $SU(2)$ group. This is particularly useful in practical phenomenological applications [22]. To do this, let

$$f^{\hat{a}} = \left(\delta^{\hat{a}\hat{b}}\partial_\mu - gf^{\hat{a}\hat{b}\bar{c}}A^{\bar{c}}_\mu\right)A^\mu_{\hat{b}} \tag{II.23}$$

be the corresponding gauge-fixing functions. In the $SU(2)$ coordinates, these functions can be arranged in a doublet of this group as follows:

$$f_Y = D_\mu Y^\mu , \tag{II.24}$$

where D_μ is the covariant derivative in the fundamental representation of $SU(2)$.

We now proceed to study the Hamiltonian structure of the theory from the $SU(2)$ point of view. So, to describe the system in phase space terms the following conjugate momenta are defined:

$$\pi_{W}{}^{\mu}_{\bar{a}} = \frac{\partial \mathcal{L}_{SU(2)}}{\partial \dot{W}^{\bar{a}}_{\mu}} = F^{\mu 0}_{\bar{a}} \ , \tag{II.25a}$$

$$\pi_{Y}{}^{\mu}_{r} = \frac{\partial \mathcal{L}_{SU(2)}}{\partial \dot{Y}^{r}_{\mu}} = Y^{*\,\mu 0}_{r} \ , \tag{II.25b}$$

$$\pi^{*\,\mu\, r}_{Y} = \frac{\partial \mathcal{L}_{SU(2)}}{\partial \dot{Y}^{*}_{\mu\, r}} = Y^{\mu 0\, r} \ , \tag{II.25c}$$

$$\pi_{Z}{}^{\mu} = \frac{\partial \mathcal{L}_{SU(2)}}{\partial \dot{Z}_{\mu}} = F^{\mu 0}_{8} \ . \tag{II.25d}$$

It is important to notice that $\pi_{W}{}^{\mu}_{\bar{a}}$ are not the canonical momenta associated with the pure $SU(2)$ theory whose Lagrangian is

$$\mathcal{L} = -\frac{1}{4} W^{\bar{a}}_{\mu\nu} W^{\mu\nu}_{\bar{a}} \ , \tag{II.26}$$

and which leads to the conjugate momenta

$$p^{\mu}_{\bar{a}} = W^{\mu 0}_{\bar{a}} \ . \tag{II.27}$$

These momenta differ from those derived from the Lagrangian (II.17), which are explicitly given by

$$\pi_{W}{}^{\mu}_{\bar{a}} = p^{\mu}_{\bar{a}} + ig\left(Y^{\dagger\mu} \frac{\sigma^{\bar{a}}}{2} Y^{0} - Y^{\dagger 0} \frac{\sigma^{\bar{a}}}{2} Y^{\mu} \right) \ . \tag{II.28}$$

The relations between the canonical momenta in the $SU(3)$ and the $SU(2)$ descriptions are

$$\pi^{\mu}_{\bar{a}} = \pi_{W}{}^{\mu}_{\bar{a}} \ , \tag{II.29a}$$

$$\pi^{\mu}_{4} = \frac{1}{\sqrt{2}}\left(\pi_{Y}{}^{\mu}_{1} + \pi^{*\,\mu 1}_{Y} \right) \ , \ \pi^{\mu}_{5} = \frac{1}{\sqrt{2}}\left(\pi_{Y}{}^{\mu}_{1} - \pi^{*\,\mu 1}_{Y} \right) \ , \tag{II.29b}$$

$$\pi^{\mu}_{6} = \frac{1}{\sqrt{2}}\left(\pi_{Y}{}^{\mu}_{2} + \pi^{*\,\mu 2}_{Y} \right) \ , \ \pi^{\mu}_{7} = \frac{1}{\sqrt{2}}\left(\pi_{Y}{}^{\mu}_{1} - \pi^{*\,\mu 1}_{Y} \right) \ , \tag{II.29c}$$

$$\pi^{\mu}_{8} = \pi_{Z} \ , \tag{II.29d}$$

whose inverses are

$$\pi_{W}{}^{\mu}_{\bar{a}} = \pi^{\mu}_{\bar{a}} \ , \tag{II.30a}$$

$$\pi_{Y}{}^{\mu} = (\pi_{Y}{}^{\mu}_{1} \ \ \pi_{Y}{}^{\mu}_{2}) = \frac{1}{\sqrt{2}}(\pi^{\mu}_{4} + i\pi^{\mu}_{5} \ \ \pi^{\mu}_{6} + i\pi^{\mu}_{7}) \ , \tag{II.30b}$$

$$\pi^{\dagger\,\mu}_{Y} = \begin{pmatrix} \pi^{*\,\mu 1}_{Y} \\ \pi^{*\,\mu 2}_{Y} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pi^{\mu}_{4} - i\pi^{\mu}_{5} \\ \pi^{\mu}_{6} - i\pi^{\mu}_{7} \end{pmatrix} \ , \tag{II.30c}$$

$$\pi_{Z}{}^{\mu} = \pi^{\mu}_{8} \ . \tag{II.30d}$$

From the conjugate momentum expressions (II.25) and using the notation (II.30), one can readily recognize the primary constraints as

$$\phi^{(1)}_{\bar{a}} = \pi_{W}{}^{0}_{\bar{a}} \approx 0 \ , \tag{II.31a}$$

$$\phi^{(1)}_{Y} = \pi_{Y}{}^{0} \approx 0 \ , \tag{II.31b}$$

$$\phi^{(1)\dagger}_{Y} = \pi^{\dagger\,0}_{Y} \approx 0 \ , \tag{II.31c}$$

$$\phi^{(1)}_{Z} = \pi_{Z}{}^{0} \approx 0 \ . \tag{II.31d}$$

Notice that $\phi^{(1)}_{Y}$ and $\phi^{(1)\dagger}_{Y}$ are covariant and contravariant SU(2) doublets, respectively. Since $\pi_{W}{}^{\mu}_{\bar{a}}$ do not coincide with the conjugate momenta associated to the pure $SU(2)$ theory (II.26), the primary constraints $\phi^{(1)}_{\bar{a}}$ differ from the

primary constraints $p^0_{\bar{a}}$ which emerge in the canonical analysis of (II.26). The same observation will apply for the secondary constraints.

The primary Hamiltonian, which governs the evolution of the system, takes the form

$$\mathcal{H}^{(1)}_{SU(2)} = \mathcal{H}_{SU(2)} + \mu^{\bar{a}}\phi^{(1)}_{\bar{a}} + \phi^{(1)}_{Y}\mu_Y + \mu^\dagger_Y\phi^{(1)\dagger}_Y + \mu_Z\phi^{(1)}_Z \ . \tag{II.32}$$

It corresponds to the sum of the canonical Hamiltonian

$$\begin{aligned}\mathcal{H}_{SU(2)} = {}& \tfrac{1}{2}\pi_W{}^i_{\bar{a}}\pi_W{}^i_{\bar{a}} + \pi_Y{}^i\pi^\dagger_Y{}^i + \tfrac{1}{2}\pi_Z{}^i\pi_Z{}^i + \tfrac{1}{4}\left(F^{\bar{a}}_{ij}F^{ij}_{\bar{a}} + 2Y^\dagger_{ij}Y^{ij} + F^8_{ij}F^{ij}_8\right) \\ & - W^{\bar{a}}_0\phi^{(2)}_{\bar{a}} - \phi^{(2)\dagger}_Y Y_0 - Y^\dagger_0\phi^{(2)}_Y - Z_0\phi^{(2)}_Z \ ,\end{aligned} \tag{II.33}$$

and a linear combination of the primary constraints (II.31) where the Lagrange multipliers μ_Y, $\mu^\dagger_Y$ and μ_Z are

$$\mu_Y = \frac{1}{\sqrt{2}}\begin{pmatrix}\mu^4 - i\mu^5 \\ \mu^6 - i\mu^7\end{pmatrix} \ , \tag{II.34a}$$

$$\mu^\dagger_Y = \frac{1}{\sqrt{2}}(\mu^4 + i\mu^5 \quad \mu^6 + i\mu^7) \ , \tag{II.34b}$$

$$\mu_Z = \mu_8 \ . \tag{II.34c}$$

By using the primary Hamiltonian (II.32), the consistency condition over the primary constraints (II.31) yields the following secondary constraints:

$$\phi^{(2)}_{\bar{a}} = \mathcal{D}^{\bar{a}\bar{b}}_i\pi_W{}^i_{\bar{b}} - ig\left(\pi_Y{}^i\frac{\sigma^{\bar{a}}}{2}Y_i - Y^\dagger_i\frac{\sigma^{\bar{a}}}{2}\pi^\dagger_Y{}^i\right) \approx 0 \ , \tag{II.35a}$$

$$\phi^{(2)\dagger}_Y = \pi_Y{}^i\left(\overleftarrow{D}_i + ig\frac{\sqrt{3}}{2}Z_i\right) - igY^\dagger_i\left(\frac{\sigma^{\bar{a}}}{2}\pi_W{}^i_{\bar{a}} + \frac{\sqrt{3}}{2}\pi_Z{}^i\right) \approx 0 \ , \tag{II.35b}$$

$$\phi^{(2)}_Y = \left(D_i - ig\frac{\sqrt{3}}{2}Z_i\right)\pi^\dagger_Y{}^i + ig\left(\frac{\sigma^{\bar{a}}}{2}\pi_W{}^i_{\bar{a}} + \frac{\sqrt{3}}{2}\pi_Z{}^i\right)Y_i \approx 0 \ , \tag{II.35c}$$

$$\phi^{(2)}_Z = \partial_i\pi_Z{}^i - ig\frac{\sqrt{3}}{2}\left(\pi_Y{}^iY_i - Y^\dagger_i\pi^\dagger_Y{}^i\right) \approx 0 \ , \tag{II.35d}$$

where the action of $\overleftarrow{D}_\mu$ on a contravariant $SU(2)$ doublet, say $\pi_Y{}^\mu$, is another contravariant $SU(2)$ doublet defined by $\pi_Y{}^\mu\overleftarrow{D}_\mu \equiv \partial_\mu\pi_Y{}^\mu + ig\pi_Y{}^\mu\frac{\sigma^{\bar{a}}}{2}W^{\bar{a}}_\mu$. The consistency condition applied to each secondary constraint yields no new constraints. It turns out, that all primary and secondary constraints do form a set of first–class constraints; in fact, the relevant Poisson brackets between these first–class constraints are

$$\{\phi^{(2)}_{\bar{a}}[u], \phi^{(2)}_{\bar{b}}[v]\}_{SU(2)} = g\epsilon_{\bar{a}\bar{b}\bar{c}}\,\phi^{(2)}_{\bar{c}}[uv] \ , \tag{II.36a}$$

$$\{\phi^{(2)}_{\bar{a}}[u], \phi^{(2)r}_Y[v]\}_{SU(2)} = ig\frac{(\sigma^{\bar{a}})^r_s}{2}\,\phi^{(2)s}_Y[uv] \ , \tag{II.36b}$$

$$\{\phi^{(2)r}_Y[u], \phi^{(2)s}_Y[v]\}_{SU(2)} = g^2T^{rs}_{pq}\int d^3x\,(uv)(\mathbf{x})\big(\pi^*_Y{}^{iq}Y^p_i - \pi^*_Y{}^{ip}Y^q_i\big)(\mathbf{x}) \ , \tag{II.36c}$$

$$\begin{aligned}\{\phi^{(2)r}_Y[u], \phi^{(2)*}_{Ys}[v]\}_{SU(2)} = {}& ig\left(\frac{(\sigma^{\bar{a}})^r_s}{2}\,\phi^{(2)}_{\bar{a}}[uv] + \frac{\sqrt{3}}{2}\delta^r_s\phi^{(2)}_Z[uv]\right) \\ & + g^2(T^{rs}_{pq} - T^{rs}_{qp})\int d^3x\,(uv)(\mathbf{x})\big(\pi_Y{}^i_sY^p_i - Y^*_{is}\pi^*_Y{}^{ip}\big)(\mathbf{x}) \ ,\end{aligned} \tag{II.36d}$$

$$\{\phi^{(2)r}_Y[u], \phi^{(2)}_Z[v]\}_{SU(2)} = -\frac{ig\sqrt{3}}{2}\phi^{(2)r}_Y[uv] \ , \tag{II.36e}$$

where $\{\cdot,\cdot\}_{SU(2)}$ denotes the Poisson bracket that involves the $SU(2)$ phase space coordinates. Due to the symmetries present in the lower indices of T^{rs}_{pq} one has that the terms proportional to g^2 on the right hand side of (II.36c) and (II.36d) do not contribute to the occurrence of tertiary constraints, instead these terms identically vanish and the Poisson brackets among all the constraints give a linear combination of constraints themselves. A more elegant

argument to show that such terms must identically vanish on the whole phase space is the following. Notice that (II.14) and (II.29) define a canonical transformation in the ordinary sense [23], and hence $\{\cdot,\cdot\}_{SU(3)} = \{\cdot,\cdot\}_{SU(2)}$. Moreover, it is easy to see that this canonical transformation maps the primary constraints (II.6) into (II.31), hence the primary Hamiltonian in the $SU(3)$ phase space coordinates (II.8) becomes the corresponding Hamiltonian in the $SU(2)$ coordinates (II.32). As a consequence, the set of secondary constraints in both formalisms must match under the canonical transformation. Indeed, this can be proved by direct calculation. Since exclusively the primary Hamiltonian is employed to evolve the constraints in time through the Poisson bracket, one concludes the Dirac algorithm in the $SU(2)$ formulation must lack of tertiary constraints just as it does in the $SU(3)$ formulation; this fact rules out the presence of the extra–terms proportional to g^2 in the gauge algebra (II.36). In conclusion the canonical transformation (II.14)–(II.29) maps each stage of the Dirac algorithm in the $SU(3)$ formulation into the corresponding stage in the $SU(2)$ one. Notice that the number of physical degrees of freedom of the $SU(2)$ effective theory matches with the corresponding number of the pure $SU(3)$ Yang-Mills theory.

We end the Hamiltonian analysis from the $SU(2)$ perspective by calculating the gauge generator G [19]. This generator is linear in all first–class constraints (II.31) and (II.35) with coefficients of the primary ones related to that of the secondary ones; the relation among the coefficients is obtained by imposing the condition that the total time derivative of G,

$$\frac{\partial G}{\partial t} + \{G, \mathcal{H}_{SU(2)}\}_{SU(2)} \ ,$$

must be a linear combination of the primary constraints only [24]. As a consequence one gets

$$\begin{aligned} G = {} & \big[\mathcal{D}_0^{\bar{a}\bar{b}}\alpha^{\bar{b}} - ig\big(\beta^\dagger\frac{\sigma^{\bar{a}}}{2}Y_0 - Y_0^\dagger\frac{\sigma^{\bar{a}}}{2}\beta\big)\big]\phi^{(1)}_{\bar{a}} + \phi^{(1)}_Y\big[\big(D_0 - ig\frac{\sqrt{3}}{2}Z^0\big)\beta + ig\big(\frac{\sigma^{\bar{a}}}{2}\alpha^{\bar{a}} - \frac{\sqrt{3}}{2}\alpha_Z\big)Y_0\big] \\ & + \big[\beta^\dagger\big(\overleftarrow{D}_0 + ig\frac{\sqrt{3}}{2}Z^0\big) - igY_0^\dagger\big(\frac{\sigma^{\bar{a}}}{2}\alpha^{\bar{a}} - \frac{\sqrt{3}}{2}\alpha_Z\big)\big]\phi^{(1)\dagger}_Y + \big[\partial_0\alpha_Z + ig\big(\beta Y_0^\dagger - \beta^\dagger Y_0\big)\big]\phi^{(1)}_Z \\ & - \alpha^{\bar{a}}\phi^{(2)}_{\bar{a}} - \beta^\dagger\phi^{(2)}_Y - \phi^{(2)\dagger}_Y\beta - \alpha_Z\phi^{(2)}_Z \ . \end{aligned} \tag{II.37}$$

This gauge generator is the sum of $G_{\rm s} \equiv G|_{\beta=0,\alpha_Z=0}$ and $G_{\rm ns} \equiv G|_{\alpha^{\bar{a}}=0}$ which independently generate the SGT and NSGT, Eqs. (II.19) and (II.20), respectively, via the Poisson brackets

$$\delta_{\rm s}W^{\bar{a}}_\mu = \{W^{\bar{a}}_\mu, G_{\rm s}\}_{SU(2)},\ \delta_{\rm s}Y_\mu = \{Y_\mu, G_{\rm s}\}_{SU(2)},\ \delta_{\rm s}Z_\mu = \{Z_\mu, G_{\rm s}\}_{SU(2)} \ , \tag{II.38a}$$
$$\delta_{\rm ns}W^{\bar{a}}_\mu = \{W^{\bar{a}}_\mu, G_{\rm ns}\}_{SU(2)},\ \delta_{\rm ns}Y_\mu = \{Y_\mu, G_{\rm ns}\}_{SU(2)},\ \delta_{\rm ns}Z_\mu = \{Z_\mu, G_{\rm ns}\}_{SU(2)} \ . \tag{II.38b}$$

The complete transformations (II.18) are duly reproduced by the addition $\delta = \delta_{\rm s} + \delta_{\rm ns}$. It is worth noticing that the gauge generator (II.37) is the image of the gauge generator (II.12) under the canonical transformation defined by (II.14) and (II.29).

To conclude this subsection, we would like to emphasize the following. A hidden symmetry arises when an admissible canonical transformation is introduced. The canonical transformation is admissible in the sense that it maps covariant objects under some group G into covariant objects of a subgroup H of G. The gauge symmetry, which is manifest in G, is hidden in H. The gauge symmetries with respect to the group G that appear hidden from the H perspective are those associated with the generators of G that do not generate H. This is true independently of whether or not the G group is spontaneously broken down into H. In our toy model $G = SU(3)$ and $H = SU(2)$, after the canonical transformation, only the fields $W^{\bar{a}}_\mu = A^{\bar{a}}_\mu$ explicitly continue being gauge fields under H. The rest of the fields, Y_μ, $Y^\dagger_\mu$ and Z_μ, fulfill very different transformation laws under H; nevertheless, the latter fields can be mapped back with the canonical transformation to gauge fields with respect to G. This result is crucial for our study of passing from the $SU(N,\mathcal{M})$ gauge group description to the $SU(N,\mathcal{M}^4)$ one via compactification, as in this case the phenomenon of spontaneous symmetry breaking is not present. Note that in this subtler case $SU(N,\mathcal{M}^4)$ is a subgroup of $SU(N,\mathcal{M})$ not due to a difference in the number of generators, which is the same indeed, but because the gauge parameters of the group $SU(N,\mathcal{M})$ are restricted to take values on the submanifold $\mathcal{M}^4$ of $\mathcal{M}$. We will show that there exists an admissible canonical transformation in this case.

III. THE $SU(3)$ YANG-MILLS THEORY WITH SPONTANEOUS SYMMETRY BREAKING

We now proceed to extend the study of the previous section to the case when the $SU(3)$ group is spontaneously broken into the $SU(2)$ in the usual sense. One of the main purposes is to show how the $SU(2)$ description is natural in this case of SSB. Although most of the material presented here is well known, we consider that its inclusion is relevant,

as it might facilitate the discussion on Yang-Mills theories with compactified extra dimensions to be introduced in the next section. One remarkable feature in the compactification scheme applied to Yang-Mills theories is the presence of the Higgs mechanism. In particular, we consider that it is important to show how the NSGT specified in the $SU(3)$ model with SSB can be used to define the unitary gauge; in this scenario, we will be able to make a precise analogy of this procedure with a similar one used in the context of extra dimensions.

A. The $SU(3)$ perspective of the model

To carry out the mentioned SSB, we add to the pure $SU(3)$ theory given by the Lagrangian (II.1) a renormalizable scalar sector $\mathcal{L}_\Phi$, so that

$$\mathcal{L}_{SU(3),\Phi} = \mathcal{L}_{SU(3)} + \mathcal{L}_\Phi \,, \tag{III.1}$$

where

$$\mathcal{L}_\Phi = (D_\mu\Phi)^\dagger(D^\mu\Phi) - V(\Phi^\dagger,\Phi)\,. \tag{III.2}$$

In this expression $D_\mu = \partial_\mu - ig\frac{\lambda^a}{2}A^a_\mu$ is the covariant derivative in the fundamental representation of $SU(3)$[1] and Φ is a complex contravariant Poincaré scalar triplet of $SU(3)$. In addition, $V(\Phi^\dagger,\Phi)$ is the renormalizable scalar potential given by

$$V(\Phi^\dagger,\Phi) = \mu^2\left(\Phi^\dagger\Phi\right) + \lambda\left(\Phi^\dagger\Phi\right)^2 \,. \tag{III.3}$$

It is straightforward to show that the Lagrangian (III.1) is simultaneously invariant under (II.3) and the infinitesimal rotation of the triplet Φ in the isospin space

$$\delta\Phi = -i\alpha^a\left(\frac{\lambda^a}{2}\Phi\right) \,. \tag{III.4}$$

The gauge symmetries of the Lagrangian (III.1) will be reflected in the occurrence of first–class constraints in the Hamiltonian setting. In order to formulate the theory in phase space terms, in addition to the canonical pairs (A^a_μ,π^μ_a), *cf.* Eqs. (II.5), the conjugate pairs (Φ,π) and $(\Phi^\dagger,\pi^\dagger)$ must be introduced; where

$$\pi = \frac{\partial\mathcal{L}_\Phi}{\partial\dot{\Phi}} = (D_0\Phi)^\dagger \,, \tag{III.5a}$$

$$\pi^\dagger = \frac{\partial\mathcal{L}_\Phi}{\partial\dot{\Phi}^\dagger} = D_0\Phi \,, \tag{III.5b}$$

Note that π and $\pi^\dagger$ correspond to covariant and contravariant $SU(3)$ triplets, respectively. From the Eqs. (III.5a) and (III.5b) the velocities $\dot{\Phi}^\dagger$ and $\dot{\Phi}$ are expressible in terms of phase space variables, therefore they do not give rise to more primary constraints in addition to those defined in (II.6). To bring uniformity into the present section, primary constraints will be denoted by $\varphi^{(1)}_a \equiv \phi^{(1)}_a$. The incorporation of the scalar sector to the pure $SU(3)$ Yang-Mills Lagrangian does not have influence upon the primary constraints of the pure theory alone.

The canonical Hamiltonian associated to (III.1) will be the sum of (II.9) and the contribution from the Higgs sector $\mathcal{L}_\Phi$, namely

$$\mathcal{H}_{SU(3),\Phi} = \mathcal{H}_{SU(3)} + \mathcal{H}_\Phi \,, \tag{III.6}$$

where

$$\mathcal{H}_\Phi = \ \pi\pi^\dagger + igA^a_0\big(\pi\frac{\lambda^a}{2}\Phi - \Phi^\dagger\frac{\lambda^a}{2}\phi^\dagger\big) - \big(D_i\Phi\big)^\dagger(D^i\Phi) + V(\Phi,\Phi^\dagger) \tag{III.7}$$

[1] We trust that no confusion will arise with the symbol D_μ already used for the covariant derivative of $SU(2)$ in its fundamental representation, as we think one can infer the nature of the covariant derivative depending on which object this is acting on.

Notice that the term linear in A_0^a will modify the secondary constraints that are produced in the absence of the Higgs sector. Indeed, the primary Hamiltonian

$$\mathcal{H}^{(1)}_{SU(3),\Phi} = \mathcal{H}_{SU(3),\Phi} + \mu^a \varphi_a^{(1)} \tag{III.8}$$

allows us to obtain the consistency condition on the primary constraints (II.6) providing the following secondary constraints:

$$\varphi_a^{(2)} \equiv \phi_a^{(2)} - ig\big(\pi \frac{\lambda^a}{2}\Phi - \Phi^\dagger \frac{\lambda^a}{2}\pi^\dagger\big) \approx 0 \ , \tag{III.9}$$

where $\phi_a^{(2)}$ corresponds to the secondary constraints (II.10) conveyed by the pure $SU(3)$ Yang-Mills theory . Consistency requirement on $\varphi_a^{(2)}$ does not bring more constraints, ending with the Dirac algorithm. The primary and secondary constraints of the theory, Eqs. (II.6) and (III.9), form a set of first–class constraints; the nonvanishing Poisson brackets between the constraints reveal the $SU(3)$ symmetry of the theory

$$\{\varphi_a^{(2)}[u], \varphi_b^{(2)}[v]\}_{SU(3)} = g f_{abc} \varphi_c^{(2)}[uv] \ , \tag{III.10}$$

where $\{\cdot,\cdot\}_{SU(3)}$ is the Poisson bracket in the $SU(3)$ formulation which takes into account the conjugate pairs (A^a_μ, π_a^μ), (Φ, π) and $(\Phi^\dagger, \pi^\dagger)$. Since only secondary constraints are modified by the Higgs sector, one expects that once the SSB of $SU(3)$ into $SU(2)$ operates, the affected constraints will only be the secondary ones.

Before going into the $SU(2)$ formulation of the theory, the gauge generator is presented. Linear in all first–class constraints, this corresponds to

$$G = (\mathcal{D}_0^{ab}\alpha^b)\varphi_a^{(1)} - \alpha^a \varphi_a^{(2)} \tag{III.11}$$

Notice that the scalar contribution in the secondary constraints (III.9) is responsible for the appropriate transformation law that the scalar fields must follow (III.4); in fact,

$$\delta A^a_\mu = \{A^a_\mu, G\}_{SU(3)} \tag{III.12a}$$
$$\delta \Phi = \{\Phi, G\}_{SU(3)} \tag{III.12b}$$

faithfully reproduce (II.3) and (III.4), that is, the symmetries of the theory.

B. SSB from the $SU(3)$ perspective

In this subsection we revisit the SSB [4] from what we have referred to as the $SU(3)$ perspective. We consider the case $\mu^2 < 0$, in which the vacuum is infinitely degenerate, so the theory presents SSB.

The extremum at $\Phi = 0$ is not considered. We may presume that the expectation value of Φ in the vacuum does not vanish. The energy of the system is minimal on all the points of the spherical surface given by

$$\Phi^\dagger_{\rm min}\Phi_{\rm min} = -\frac{\mu^2}{2\lambda} \equiv v^2 \ , \tag{III.13}$$

All points on these surface are physically equivalent because they are connected through $SU(3)$ transformations. To break down $SU(3)$ into $SU(2)$, one chooses a particular direction $\Phi_{\rm min}$ such that

$$\frac{\lambda^{\bar a}}{2}\Phi_{\rm min} = 0 \ , \tag{III.14a}$$
$$\frac{\lambda^{\hat a}}{2}\Phi_{\rm min} \neq 0 \ , \tag{III.14b}$$
$$\frac{\lambda^{8}}{2}\Phi_{\rm min} \neq 0 \ . \tag{III.14c}$$

The isotropy group, the one corresponding to unbroken symmetries, at $\Phi_{\rm min}$ is $SU(2)$. It is convenient to choose a representative of the solutions (III.13) as $\Phi^\dagger_{\rm min} = (0 \ \ 0 \ \ v)$. This choice means that five generators of $SU(3)$, namely, $\frac{\lambda^{\hat a}}{2}$ and $\frac{\lambda^8}{2}$, are broken.

Within this formulation, two cases naturally arise depending on the nature of the gauge parameters α^a (*cf.* (II.3)). These are

(*i*) *The Goldstone Theorem* [4]. Assuming the parameters α^a to be constant functions on Minkowski space, the invariant Lagrangian corresponds to

$$\mathcal{L}_{SU(3),H} = (\partial_\mu \Phi)^\dagger (\partial^\mu \Phi) - V(\Phi^\dagger, \Phi) \ .$$

When the theory is subjected to the translation $\Phi \mapsto \varphi \equiv \Phi - \Phi_{\min}$ there arise five real massless scalars. These correspond to φ^1, φ^2 and the imaginary part of φ^3 denoted as ϕ_Z. In addition, a massive scalar H emerges, identified as the real part of φ^3, that quantifies the normal excitations to the surface of the minimal energy. Hence, associated with each broken generator of $SU(3)$ there is a massless scalar or Goldstone boson.

(*ii*) *The Higgs Mechanism* [3]. Assuming the parameters α^a to be nonconstant functions on Minkowski space, the invariant Lagrangian corresponds to (II.1). In this case, besides the presence of five Goldstone bosons, five massive gauge bosons ($A^{\hat{a}}_\mu$ and A^8_μ) arise. This is the celebrated Higgs mechanism. In this scenario, the Goldstone bosons represent spurious degrees of freedom, as they can be removed from the theory in a special gauge, known as unitary gauge. In the following section we will show that this mechanism has a natural description in the $SU(2)$ coordinates, and that the unitary gauge can be understood as the action of fixing the parameters within what will be defined as NSGT on the scalar fields, Eq. (III.21b).

C. The $SU(2)$ perspective of the model

In this subsection the description of the field theory (III.1) from the $SU(2)$ perspective is achieved. The pure $SU(3)$ Yang-Mills sector $\mathcal{L}_{SU(3)}$ is mapped, by means of the point transformation (II.14), into $\mathcal{L}_{SU(2)}$ (II.17), and the scalar sector $\mathcal{L}_\Phi$ is mapped into $\mathcal{L}_\phi$ by decomposing the $SU(3)$ triplet Φ into a $SU(2)$ doublet and a scalar,

$$\begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} = \begin{pmatrix} \Phi^1 \\ \Phi^2 \end{pmatrix}, \tag{III.15a}$$

$$\phi^0 = \Phi^3 \ . \tag{III.15b}$$

Therefore, the Lagrangian (III.1) is recast in terms of well defined objects under the action of $SU(2)$,

$$\mathcal{L}_{SU(2),\phi} = \mathcal{L}_{SU(2)} + \mathcal{L}_\phi, \tag{III.16}$$

where the Higgs sector becomes

$$\mathcal{L}_\phi = (D_\mu \Phi)^\dagger (D^\mu \Phi)\big|_{\substack{\Phi \to \phi \\ A_\mu \to W_\mu, Y_\mu, Y^\dagger_\mu, Z_\mu}} + V(\Phi, \Phi^\dagger)\big|_{\Phi \to \phi} \ . \tag{III.17}$$

The first term explicitly becomes

$$\begin{aligned} &(D_\mu \phi)^\dagger (D^\mu \phi) + (\partial_\mu \phi^{0*})(\partial^\mu \phi^0) + \frac{g^2}{2}\left[(\phi^{0*}\phi^0)(Y^\dagger_\mu Y^\mu) + (\phi^\dagger Y^\mu)(Y^\dagger_\mu \phi)\right] \\ &+ \frac{ig}{\sqrt{2}}\left[\phi^{0*} Y^\dagger_\mu (D^\mu \phi) - \phi^0 (D^\mu \phi)^\dagger Y_\mu + (\phi^\dagger Y_\mu)(\partial^\mu \phi^0) - (Y^\dagger_\mu \phi)(\partial^\mu \phi^{0*})\right] \\ &+ \frac{g^2}{12} Z_\mu Z^\mu \left(4\phi^{0*}\phi^0 + \phi^\dagger \phi\right) - \frac{g^2}{2\sqrt{6}} Z_\mu \left[\phi^0 (\phi^\dagger Y^\mu) + \phi^{0*} (Y^{\mu\dagger} \phi)\right] \\ &+ \frac{ig}{2\sqrt{3}} Z_\mu \left[\phi^\dagger (D^\mu \phi) - (D^\mu \phi)^\dagger \phi + 2\left(\phi^0 \partial^\mu \phi^{0*} - \phi^{0*} \partial^\mu \phi^0\right)\right] \ , \end{aligned} \tag{III.18}$$

where the covariant derivative in this expression is the one associated to the $SU(2)$ group in the fundamental representation. The scalar potential can be written as follows:

$$V|_{\Phi \to \phi} = \left[\mu^2 + 2\lambda\left(\phi^{0*}\phi^0\right)\right]\left(\phi^\dagger \phi\right) + \left[\mu^2 + \lambda\left(\phi^{0*}\phi^0\right)\right]\left(\phi^{0*}\phi^0\right) + \lambda\left(\phi^\dagger \phi\right)^2 \ . \tag{III.19}$$

Gauge invariances of the theory in this formulation correspond to (III.1) together with

$$\delta\phi = -i\left(\frac{\sigma^{\bar{a}}}{2}\alpha^{\bar{a}} + \frac{1}{2\sqrt{3}}\alpha_Z\right)\phi - \frac{i}{\sqrt{2}}\phi^0 \beta \ , \tag{III.20a}$$

$$\delta\phi^0 = -\frac{i}{\sqrt{2}}\beta^\dagger \phi + \frac{i}{\sqrt{3}}\alpha_Z \phi^0 \ . \tag{III.20b}$$

Notice that in the scalar sector of the theory, the SGT and NSGT also arise naturally. Indeed

$$\delta_{\rm s}\phi = -i\frac{\sigma^{\bar a}}{2}\alpha^{\bar a}\phi\,,\ \ \delta_{\rm s}\phi^0 = 0\ ; \tag{III.21a}$$

$$\delta_{\rm ns}\phi = -\frac{i}{2}\left(\frac{1}{\sqrt{3}}\alpha_Z\phi + \sqrt{2}\phi^0\beta\right)\ ,\ \ \delta_{\rm ns}\phi^0 = -\frac{i}{\sqrt{2}}\beta^\dagger\phi + \frac{i}{\sqrt{3}}\alpha_Z\phi^0\ . \tag{III.21b}$$

We now proceed to the Hamiltonian formulation associated to the singular Lagrangian (III.16). Since the scalar sector does not contain spacetime derivatives of either gauge fields $W^{\bar a}_\mu$, or $SU(2)$ doublets Y_μ, or the scalar Z_μ, the canonical conjugate momentum associated to each of these fields coincides with those defined in Sec. II B. Hence the conjugate momenta in the $SU(2)$ formulation are given by Eqs. (II.25) and

$$\pi_\phi = \frac{\partial\mathcal{L}_\phi}{\partial\dot\phi} = \phi^\dagger\left(\overleftarrow{D}_0 + \frac{ig}{2\sqrt{3}}Z_0\right) + \frac{ig}{\sqrt{2}}\phi^{0\,*}\,Y_0^\dagger\ , \tag{III.22a}$$

$$\pi_0 = \frac{\partial\mathcal{L}_\phi}{\partial\dot\phi^0} = \left(\partial_0 - \frac{ig}{\sqrt{3}}Z_0\right)\phi^{0\,*} + \frac{ig}{\sqrt{2}}\phi^\dagger\,Y_0\ , \tag{III.22b}$$

$$\pi_\phi^\dagger = \frac{\partial\mathcal{L}_\phi}{\partial\dot\phi^\dagger} = \left(D_0 - \frac{ig}{2\sqrt{3}}Z_0\right)\phi - \frac{ig}{\sqrt{2}}\phi^0\,Y_0\ , \tag{III.22c}$$

$$\pi_0^* = \frac{\partial\mathcal{L}_\phi}{\partial\dot\phi^{0\,*}} = \left(\partial_0 + \frac{ig}{\sqrt{3}}Z_0\right)\phi^0 - \frac{ig}{\sqrt{2}}Y_0^\dagger\,\phi\ . \tag{III.22d}$$

It is worth noticing that π_ϕ and $\pi_\phi^\dagger$ are covariant and contravariant $SU(2)$ doublets, respectively; whereas, π_0 and its complex conjugate are $SU(2)$ scalars. The relations among conjugate momenta (III.22) and the corresponding objects (III.5) in the $SU(3)$ viewpoint are

$$\pi_\phi = (\pi_\phi^1\ \ \pi_\phi^2) = (\pi^1\ \ \pi^2)\ , \tag{III.23a}$$

$$\pi_0 = \pi^3\ . \tag{III.23b}$$

As expected, the scalar sector of the theory does not bring additional constraints into the $SU(2)$ formalism either. Instead of going through the Dirac formalism using the Poisson bracket $\{\cdot,\cdot\}_{SU(2)}$, that in this case would include also the canonical pairs (ϕ,π_ϕ) and (ϕ^0,π_0), we will make use of the arguments given after Eqs. (II.38) in the following way. First, notice that Eqs. (II.14), (II.29), (III.15) and (III.23) define a canonical transformation from $SU(3)$ to $SU(2)$ coordinates, therefore $\{\cdot,\cdot\}_{SU(3)} = \{\cdot,\cdot\}_{SU(2)}$. Second, the canonical transformation maps the set of primary constraints $\{\varphi_a^{(1)}\}$ into the set of primary constraints $\{\varphi_{\bar a}^{(1)} \equiv \phi_{\bar a}^{(1)}, \varphi_Y^{(a)} \equiv \phi_Y^{(1)}, \varphi_Z^{(1)} \equiv \phi_Z^{(1)}\}$, the transformation hence recasts the primary Hamiltonian (III.8) in terms of $SU(2)$ variables as follows:

$$\mathcal{H}^{(1)}_{SU(2),\phi} = \mathcal{H}_{SU(2)} + \mathcal{H}_\phi + \mu^{\bar a}\varphi_{\bar a}^{(1)} + \varphi_Y^{(1)}\mu_Y + \mu_Y^\dagger\varphi_Y^{(1)\dagger} + \mu_Z\varphi_Z^{(1)} \tag{III.24}$$

where $\mathcal{H}_{SU(2)}$ is given by (II.17) and $\mathcal{H}_\phi$ is the Legendre transformation of $\mathcal{L}_\phi$ with respect to the fields ϕ and ϕ^0. As a consequence of these two observations, the set of secondary constraints that emerges in the $SU(3)$ viewpoint must be faithfully mapped into the set of secondary constraints given in terms of the $SU(2)$ coordinates. These are

$$\varphi_{\bar a}^{(2)} = \phi_{\bar a}^{(2)} - ig\left(\pi_\phi\frac{\sigma^{\bar a}}{2}\phi - \phi^\dagger\frac{\sigma^{\bar a}}{2}\pi_\phi^\dagger\right) \approx 0\ , \tag{III.25a}$$

$$\varphi_Y^{(2)\dagger} = \phi_Y^{(2)\dagger} + \frac{ig}{\sqrt{2}}\left(\pi_0^*\phi^\dagger - \phi^0\pi_\phi\right) \approx 0\ , \tag{III.25b}$$

$$\varphi_Y^{(2)} = \phi_Y^{(2)} - \frac{ig}{\sqrt{2}}\left(\pi_0\phi - \phi^{0\,*}\pi_\phi^\dagger\right) \approx 0\ , \tag{III.25c}$$

$$\varphi_Z^{(2)} = \phi_Z^{(2)} - \frac{ig}{\sqrt{3}}\left(\phi^{0\,*}\pi_0^* - \pi_0\phi^0 + \frac{1}{2}(\pi_\phi\phi - \phi^\dagger\pi_\phi^\dagger)\right) \approx 0\ , \tag{III.25d}$$

where $\phi_{\bar a}^{(2)}$, $\phi_Y^{(2)\dagger}$, $\phi_Y^{(2)}$ and $\phi_Z^{(2)}$ are given by Eqs. (II.35). Indeed this can be proved by direct calculation. Finally, the set of equations that define the gauge algebra (III.10) can be expressed in terms of $SU(2)$ variables only using the

canonical transformation. The nonvanishing Poisson brackets are

$$\{\varphi^{(2)}_{\bar a}[u],\varphi^{(2)}_{\bar b}[v]\}_{SU(2)} = g\epsilon_{\bar a\bar b\bar c}\,\varphi^{(2)}_{\bar c}[uv]\ , \tag{III.26a}$$

$$\{\varphi^{(2)}_{\bar a}[u],\varphi^{(2)r}_{Y}[v]\}_{SU(2)} = ig\frac{(\sigma^{\bar a})^r_{\ s}}{2}\,\varphi^{(2)s}_{Y}[uv]\ , \tag{III.26b}$$

$$\{\varphi^{(2)r}_{Y}[u],\varphi^{(2)*}_{Ys}[v]\}_{SU(2)} = ig\left(\frac{(\sigma^{\bar a})^r_{\ s}}{2}\,\varphi^{(2)}_{\bar a}[uv] + \frac{\sqrt{3}}{2}\delta^r_s\varphi^{(2)}_{Z}[uv]\right) \tag{III.26c}$$

$$\{\varphi^{(2)r}_{Y}[u],\varphi^{(2)}_{Z}[v]\}_{SU(2)} = -\frac{ig\sqrt{3}}{2}\varphi^{(2)r}_{Y}[uv]\ , \tag{III.26d}$$

Since the canonical transformation connects the Dirac algorithm developed in the two different set of coordinates at each step of it, we have that the gauge generator (III.11) must be translated into the corresponding one in the $SU(2)$ variables, namely

$$\begin{aligned} G = {} & \big[\mathcal{D}^{\bar a\bar b}_0\alpha^{\bar b} - ig\big(\beta^\dagger\frac{\sigma^{\bar a}}{2}Y_0 - Y^\dagger_0\frac{\sigma^{\bar a}}{2}\beta\big)\big]\varphi^{(1)}_{\bar a} + \varphi^{(1)}_{Y}\big[\big(D_0 - ig\frac{\sqrt{3}}{2}Z^0\big)\beta + ig\big(\frac{\sigma^{\bar a}}{2}\alpha^{\bar a} - \frac{\sqrt{3}}{2}\alpha_Z\big)Y_0\big] \\ & + \big[\beta^\dagger\big(\overleftarrow{D}_0 + ig\frac{\sqrt{3}}{2}Z^0\big) - igY^\dagger_0\big(\frac{\sigma^{\bar a}}{2}\alpha^{\bar a} - \frac{\sqrt{3}}{2}\alpha_Z\big)\big]\varphi^{(1)\dagger}_{Y} + \big[\partial_0\alpha_Z + ig\big(\beta Y^\dagger_0 - \beta^\dagger Y_0\big)\big]\varphi^{(1)}_{Z} \\ & - \alpha^{\bar a}\varphi^{(2)}_{\bar a} - \beta^\dagger\varphi^{(2)}_{Y} - \varphi^{(2)\dagger}_{Y}\beta - \alpha_Z\varphi^{(2)}_{Z}\ . \end{aligned} \tag{III.27}$$

from which the sectors that independently generate SGT, $G_{\rm s} \equiv G|_{\beta=0,\alpha_Z=0}$, and NGST, $G_{\rm ns} \equiv G|_{\alpha^{\bar a}=0}$, are easily identified. Notice that it is due to the terms depending on the Higgs sector in each secondary constraint that Eqs. (III.21) are suitably recovered from brackets

$$\delta_{\rm s}\phi = \{\phi, G_{\rm s}\}_{SU(2)},\ \ \delta_{\rm s}\phi^0 = \{\phi^0, G_{\rm s}\}_{SU(2)}\ , \tag{III.28a}$$

$$\delta_{\rm ns}\phi = \{\phi, G_{\rm ns}\}_{SU(2)},\ \ \delta_{\rm ns}\phi^0 = \{\phi^0, G_{\rm ns}\}_{SU(2)}\ . \tag{III.28b}$$

The corresponding variations for $W^{\bar a}_\mu$, Y_μ and Z_μ are given in Eqs. (II.38).

In this subsection we have passed from a description in which the $SU(3)$ symmetry is manifest to an equivalent formulation that is manifestly invariant only under the subgroup $SU(2)$. In the context of theories with SSB, it is said that the $SU(2)$ symmetry is exact, whereas the $SU(3)$ is hidden. We now turn to discuss the SSB of the $SU(3)$ group into the $SU(2)$ one, from the viewpoint of the latter.

D. SSB from the $SU(2)$ perspective

We reconsider the case of infinite degeneracy of vacuum, $\mu^2 < 0$. Configurations with minimal energy (III.13) lie on $\phi^\dagger_{\rm min}\phi_{\rm min} + \phi^{0\,*}_{\rm min}\phi^0_{\rm min} = v^2$. As we have remarked, there is a natural separation of $SU(3)$ parameters into those parameters of the isotropy group, $\alpha^{\bar a}$, and those associated to the broken part of the group, $\alpha^{\hat a}$ and α^8. In fact, this split is what determines the SGT and NSGT previously defined. The functional form of the Lagrangian (III.16), where the $SU(2)$ sector of $SU(3)$ is manifest, suggests the study of the following cases:

(i) *The Goldstone theorem.* We assume the broken part of $SU(3)$, generated by $\frac{\lambda^{\hat a}}{2}$ and $\frac{\lambda^8}{2}$, to be global; that is, we allow $\alpha^{\hat a}$ and α^8 to be spacetime independent. In other words, assume that the NGST are global, but not necessarily SGT ones. In such a situation, the following Lagrangian is invariant under this class of transformations:

$$\mathcal{L}_g = -\frac{1}{4}W^{\bar a}_{\mu\nu}W^{\mu\nu}_{\bar a} + (D_\mu\phi)^\dagger(D^\mu\phi) + (\partial_\mu\phi^{0*})(\partial^\mu\phi^0) + V|_{\Phi\to\phi}\ ,$$

where $W^{\bar a}_{\mu\nu}$ are the components of the $su(2)$–valued curvature and D_μ is the covariant derivative of $SU(2)$ in the fundamental representation. There arise five massless scalars when the theory is developed around the particular minimum $\Phi_{\rm min}$, which is decomposed into the doublet $\phi_{\rm min} = 0$ and the scalar $\phi^0_{\rm min} = v$, by carrying out the shift $\phi^0 \mapsto H + i\phi_Z \equiv \phi^0 - v$. These scalars do correspond to ϕ, $\phi^\dagger$ and the singlet ϕ_Z, which are identified with the so–called Goldstone bosons. The massive field H survives. Hence, there is a massless scalar associated with each independent NSGT.

(*ii*) *The Higgs mechanism.* Now assume the larger symmetry $SU(3)$, that is, both the SGT and NSGT are local. In this scenario, the theory developed around the particular minimum is characterized by the Lagrangian given in Eq. (III.16), with ϕ^0 replaced by $(v + H + i\phi_Z)$. Five gauge fields, Y_μ, $Y_\mu^\dagger$, and Z_μ, acquire mass and simultaneously five pseudo–Goldstone bosons appear, namely ϕ, $\phi^\dagger$ and ϕ_Z. Notice that all the mass terms are invariant under the $SU(2)$ subgroup.

All pseudo–Goldstone bosons can be removed from the theory through the so–called unitary gauge. Although in this gauge the pseudo–Goldstone bosons disappear from the theory, the degrees of freedom that they represent appear as the longitudinal polarization states of the gauge bosons associated with the broken generators. The implementation of the unitary gauge can be understood in terms of the NSGT. Indeed, consider the NSGT (III.21b) with particular gauge parameters

$$\beta = -\frac{i\sqrt{2}}{v}\phi \ , \tag{III.29a}$$

$$\alpha_Z = -\frac{\sqrt{3}}{v}\phi_Z \ , \tag{III.29b}$$

which yields $\phi' = 0$ and $\phi'_Z = 0$. Therefore *the unitary gauge corresponds to a particular NSGT which maps the pseudo Goldstone bosons into zero.* In addition, from the NSGT given by Eqs. (II.20), one finds

$$W'^{\bar{a}}_\mu = W^{\bar{a}}_\mu \ , \tag{III.30a}$$

$$Y'_\mu = Y_\mu - \frac{i\sqrt{2}}{v}\partial_\mu\phi \ , \tag{III.30b}$$

$$Z'_\mu = Z_\mu - \frac{\sqrt{3}}{v}\partial_\mu\phi_Z \ . \tag{III.30c}$$

The incorporation of the pseudo Goldstone bosons as the longitudinal component of the massive gauge bosons Y'_μ and Z'_μ is evident from these expressions. We will come back to this latter on, when discussing this mechanism in the context of theories with compactified extra dimensions.

The unitary gauge can also be implemented via a finite NSGT. Consider the non–linear parametrization of the triplet Φ,

$$\Phi(x) = \mathbf{U}(x)\begin{pmatrix} 0 \\ 0 \\ v+H \end{pmatrix} \ , \tag{III.31}$$

with

$$\begin{aligned}\mathbf{U}(x) &= \exp\left(i\frac{\lambda^{\hat{a}}}{2}\alpha^{\hat{a}} + i\frac{\lambda^8}{2}\alpha^8\right) \\ &= \exp\Big\{-\left(\frac{i}{2v}\right)\Big[i\lambda^4\left(\phi^1 - \phi^{1*}\right) - \lambda^5\left(\phi^1 + \phi^{1*}\right) \\ &\quad + i\lambda^6\left(\phi^2 - \phi^{2*}\right) - \lambda^7\left(\phi^2 + \phi^{2*}\right) + \sqrt{\frac{3}{2}}\lambda^8\phi_Z\Big]\Big\} \ ,\end{aligned} \tag{III.32}$$

where the parameter values given in (III.29) were used. The finite version of the NSGT (III.21b) are obtained by acting with $\mathbf{U}^{-1}(x)$ as follows:

$$\Phi'(x) = \mathbf{U}^{-1}(x)\Phi = \begin{pmatrix} 0 \\ 0 \\ v+H \end{pmatrix} \ . \tag{III.33}$$

Equations (III.30) are recovered by using the particular element $\mathbf{U}^{-1}(x) \in SU(3)$ into the finite gauge transformation of the connection, $A'_\mu = U(x)A_\mu U^\dagger(x) - i(\partial_\mu U)U^\dagger$, and keeping the analysis at first order.

IV. YANG-MILLS THEORIES WITH COMPACTIFIED EXTRA DIMENSIONS

This section is devoted to study the gauge structure of Yang-Mills theories with compactified extra dimensions in the spirit of the notion of hidden symmetry discussed in previous sections. The starting point in the compactification

is Yang-Mills theory with an underlying compact Lie group $SU(N)$ in $1+(m-1)$ spacetime dimensions, with $m>4$, where we choose a metric signature that is mostly negative. We assume the spacetime manifold to be $\mathcal{M}^m = \mathcal{M}^4 \times \mathcal{N}^n = \{(X^M)\} = \{(x^\mu, y^\alpha)\} \equiv \{(x,y)\}$. To label points in the submanifold $\mathcal{N}^n$, we also use $x^{\bar{\mu}} \equiv X^{\alpha+4} = y^\alpha$, with $\alpha = 1,\dots,n$, so that, $\bar{\mu} = 5,\dots,n+4$. Gauge fields will be denoted by $\mathcal{A}^a_M(x,y)$, where a and M are the gauge ($a=1,\dots,N^2-1$) and the Lorentz ($M=0,1,2,3,5,\dots,m$) indices, respectively. Since, in a local symmetry, gauge parameters are spacetime–dependent functions, we refer to the m dimensional $SU(N)$ pure Yang-Mills theory as pure $SU(N,\mathcal{M})$ Yang-Mills. In this context, once the extra dimensions are integrated out, the effective theory will be invariant under the so–called SGT and NSGT. The former will be referred to as the $SU(N,\mathcal{M}^4)$ subgroup.

In previous sections, we introduced transformations that map covariant objects under a gauge group G into covariant objects under a subgroup H. All these maps were defined within the same standard Poincaré group $ISO(1,3)$. However, in the case of Yang-Mills theories with compact extra dimensions, the transition from the $SU(N,\mathcal{M})$ gauge group description to $SU(N,\mathcal{M}^4)$ will simultaneously convey certain transformation that maps covariant objects under the Poincaré group $ISO(1,3+n)$ into covariant objects under the standard $ISO(1,3)$. We now proceed to present a brief discussion on this issue.

A. The Poincaré group perspective

In a flat m–dimensional spacetime, the Poincaré group $ISO(1,m-1)$ is defined through its generators, whose number is equal to $\frac{1}{2}m(m+1)$. m of these generators, denoted by P_M, belong to the group of translations, and the remainders $\frac{1}{2}m(m-1)$ ones, denoted by J_{MN}, are associated with the Lorentz group $SO(1,m-1)$. These generators satisfy the following Poincaré algebra:

$$[P_M\,,\,P_N] = 0\,, \tag{IV.1}$$
$$[J_{MN}\,,\,P_R] = i\left(g_{MR}P_N - g_{NR}P_M\right)\,, \tag{IV.2}$$
$$[J_{MN}\,,\,J_{RS}] = i\left(g_{MR}J_{NS} - g_{MS}J_{NR} - g_{NR}J_{MS} + g_{NS}J_{MR}\right)\,. \tag{IV.3}$$

It is not difficult to see that in this algebra there are two subalgebras merged. One of these algebras corresponds to the one of the standard Poincaré group $ISO(1,3)$,

$$[P_\mu\,,\,P_\nu] = 0\,, \tag{IV.4}$$
$$[J_{\mu\nu}\,,\,P_\rho] = i\left(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu\right)\,, \tag{IV.5}$$
$$[J_{\mu\nu}\,,\,J_{\rho\sigma}] = i\left(g_{\mu\rho}J_{\nu\sigma} - g_{\mu\sigma}J_{\nu\rho} - g_{\nu\rho}J_{\mu\sigma} + g_{\nu\sigma}J_{\mu\rho}\right)\,, \tag{IV.6}$$

whereas the other one is associated with the inhomogeneous orthogonal group in n dimensions $ISO(n)$,

$$[P_{\bar{\mu}}\,,\,P_{\bar{\nu}}] = 0\,, \tag{IV.7}$$
$$[J_{\bar{\mu}\bar{\nu}}\,,\,P_{\bar{\rho}}] = i\left(\delta_{\bar{\nu}\bar{\rho}}P_{\bar{\mu}} - \delta_{\bar{\mu}\bar{\rho}}P_{\bar{\nu}}\right)\,, \tag{IV.8}$$
$$[J_{\bar{\mu}\bar{\nu}}\,,\,J_{\bar{\rho}\bar{\sigma}}] = i\left(\delta_{\bar{\mu}\bar{\sigma}}J_{\bar{\nu}\bar{\rho}} - \delta_{\bar{\mu}\bar{\rho}}J_{\bar{\nu}\bar{\sigma}} - \delta_{\bar{\nu}\bar{\sigma}}J_{\bar{\mu}\bar{\rho}} + \delta_{\bar{\nu}\bar{\rho}}J_{\bar{\mu}\bar{\sigma}}\right)\,. \tag{IV.9}$$

Consider an infinitesimal Poincaré transformation in the $\mathcal{M}$ manifold,

$$X^M = \omega^{MN}X_N + \epsilon^M\,, \tag{IV.10}$$

where $\omega^{MN} = -\omega^{NM}$ and ϵ^M are the infinitesimal parameters of the group. Then, under this transformation, the gauge field $\mathcal{A}_M(X) = T^a\mathcal{A}^a_M(X)$ transforms as

$$\delta\mathcal{A}_M(X) = \left[\omega_{MN} + g_{MN}\left(\omega_{AB}X^B + \epsilon_A\right)\partial^A\right]\mathcal{A}^N(X)\,. \tag{IV.11}$$

This relation can be naturally split into variations for $\mathcal{A}_\mu(X)$ and $\mathcal{A}_{\bar{\mu}}(X)$ components as follows:

$$\begin{aligned}\delta\mathcal{A}_\mu(X) &= \left[\omega_{\mu\nu} + g_{\mu\nu}\left(\omega_{\rho\sigma}x^\sigma + \epsilon_\rho\right)\partial^\rho\right]\mathcal{A}^\nu(X)\\ &+ \left[\left(\omega_{\bar{\mu}\bar{\nu}}x^{\bar{\nu}} + \epsilon_{\bar{\mu}}\right)\partial^{\bar{\mu}} + \omega_{\mu\bar{\nu}}\left(x^{\bar{\nu}}\partial^\mu - x^\mu\partial^{\bar{\nu}}\right)\right]\mathcal{A}_\mu(X)\\ &+ \omega_{\mu\bar{\nu}}\mathcal{A}^{\bar{\nu}}(X)\,,\end{aligned} \tag{IV.12}$$
$$\begin{aligned}\delta\mathcal{A}_{\bar{\mu}}(X) &= \left[\omega_{\bar{\mu}\bar{\nu}} + g_{\bar{\mu}\bar{\nu}}\left(\omega_{\bar{\rho}\bar{\sigma}}x^{\bar{\sigma}} + \epsilon_{\bar{\rho}}\right)\partial^{\bar{\rho}}\right]\mathcal{A}^{\bar{\nu}}(X)\\ &+ \left[\left(\omega_{\mu\nu}x^\nu + \epsilon_\mu\right)\partial^\mu + \omega_{\mu\bar{\nu}}\left(x^{\bar{\nu}}\partial^\mu - x^\mu\partial^{\bar{\nu}}\right)\right]\mathcal{A}_{\bar{\mu}}(X)\\ &+ \omega_{\bar{\mu}\nu}\mathcal{A}^\nu(X)\,.\end{aligned} \tag{IV.13}$$

It can be appreciated from these expressions that $\mathcal{A}_\mu$ and $\mathcal{A}_{\bar{\mu}}$ transform under the standard Lorentz group $SO(1,3)$ as a vector and as a scalar, respectively; whereas they transform as a scalar and as a vector under the orthogonal group $SO(n)$. This means that in a stage previous to compactification the Yang-Mills action $S[\mathcal{A}_M]$, which is manifestly invariant under the largest $ISO(1,m-1)$ Poincaré group, can be written in terms of covariant objects of the $ISO(1,3)$ and $ISO(n)$ groups, that is, we can write a completely equivalently theory through an action $S[\mathcal{A}_\mu, \mathcal{A}_{\bar{\mu}}]$. From this perspective, the $ISO(1,3)$ and $ISO(n)$ symmetries are manifest, but the $ISO(1,m)$ is hidden. In complete analogy with the ideas introduced in previous sections for unitary gauge groups, we can define two types of standard transformations, which correspond to the inhomogeneous subgroups $ISO(1,3)$ and $ISO(n)$. The former, which we will call standard Poincaré transformations (SPT), are defined by setting $\omega_{\bar{\mu}\bar{\nu}} = \omega_{\mu\bar{\nu}} = \epsilon_{\bar{\mu}} = 0$, which means that

$$\begin{aligned}\delta\mathcal{A}_\mu(X) &= \left[\omega_{\mu\nu} + g_{\mu\nu}\left(\omega_{\rho\sigma}x^\sigma + \epsilon_\rho\right)\partial^\rho\right]\mathcal{A}^\nu(X)\ , && \text{(IV.14)}\\ \delta\mathcal{A}_{\bar{\mu}}(X) &= \left(\omega_{\mu\nu}x^\nu + \epsilon_\mu\right)\partial^\mu\mathcal{A}_{\bar{\mu}}(X)\,. && \text{(IV.15)}\end{aligned}$$

The latter ones, which we will call standard orthogonal transformations (SOT), arise in a scenario with $\omega_{\mu\nu} = \omega_{\mu\bar{\nu}} = \epsilon_\mu = 0$. The corresponding transformations are given by

$$\begin{aligned}\delta\mathcal{A}_\mu(X) &= \left(\omega_{\bar{\mu}\bar{\nu}}x^{\bar{\nu}} + \epsilon_{\bar{\mu}}\right)\partial^{\bar{\mu}}\mathcal{A}_\mu(X)\ , && \text{(IV.16)}\\ \delta\mathcal{A}_{\bar{\mu}}(X) &= \left[\omega_{\bar{\mu}\bar{\nu}} + g_{\bar{\mu}\bar{\nu}}\left(\omega_{\bar{\rho}\bar{\sigma}}x^{\bar{\sigma}} + \epsilon_{\bar{\rho}}\right)\partial^{\bar{\rho}}\right]\mathcal{A}^{\bar{\nu}}(X)\,. && \text{(IV.17)}\end{aligned}$$

The action $S[\mathcal{A}_\mu, \mathcal{A}_{\bar{\mu}}]$ is manifestly invariant under these standard spacetime transformations. However, this action is not manifestly invariant under transformations induced by the $J_{\mu\bar{\nu}}$ generators. These are nonstandard Poincaré transformations (NSPT), which are defined in a scenario with parameters $\omega_{\mu\bar{\nu}} \neq 0$ and the remaining ones equal to zero,

$$\begin{aligned}\delta\mathcal{A}_\mu(X) &= \omega_{\rho\bar{\nu}}\left(x^{\bar{\nu}}\partial^\mu - x^\mu\partial^{\bar{\nu}}\right)\mathcal{A}_\mu(X) + \omega_{\mu\bar{\nu}}\mathcal{A}^{\bar{\nu}}(X)\,, && \text{(IV.18)}\\ \delta\mathcal{A}_{\bar{\mu}}(X) &= \omega_{\rho\bar{\nu}}\left(x^{\bar{\nu}}\partial^\rho - x^\rho\partial^{\bar{\nu}}\right)\mathcal{A}_{\bar{\mu}}(X) + \omega_{\bar{\mu}\nu}\mathcal{A}^\nu(X)\,. && \text{(IV.19)}\end{aligned}$$

As it is well known, the standard Higgs mechanism operates via scalar pseudo–Goldstone bosons that are provided by SSB of the gauge group in consideration. However, in Yang-Mills theories with compact extra dimensions the switch from the gauge group $SU(N,\mathcal{M})$ to $SU(N,\mathcal{M}^4)$ does not involve any SSB because the number of generators in both groups is the same. So, in this class of theories the pseudo–Goldstone bosons needed to implement the Higgs mechanism have nothing to do with the unitary gauge group, but instead with the Poincaré group, because these scalar fields arise by compactification of the extra spatial coordinates. Compactification leads to an explicit breaking of the $ISO(1,m)$ group into the standard one $ISO(1,3)$. So, after compactification, the corresponding effective theory, which depends on the KK fields, is subject to satisfy only the SPT.

B. Pure $SU(N,\mathcal{M})$ Yang-Mills Theory

The Lagrangian that describes pure $SU(N,\mathcal{M})$ Yang-Mills theory is given by (*cf.* (II.1))

$$\mathcal{L}_{SU(N,\mathcal{M})}(x,y) = -\frac{1}{4}\mathcal{F}^a_{MN}(x,y)\mathcal{F}^{MN}_a(x,y)\ . \tag{IV.20}$$

The components $\mathcal{F}^a_{MN}$ are regarded as functions of gauge fields $\mathcal{A}^a_M(x,y)$ as in Eq. (II.2) except that in this case the coupling constant is denoted by g_m whose dimension is of $[\text{mass}]^{(4-m)/2}$. Gauge invariances of the theory (IV.20) are (*cf.* (II.3))

$$\delta\mathcal{A}^a_M = \mathcal{D}^{ab}_M\alpha^b(x,y)\ , \tag{IV.21}$$

where $\mathcal{D}^{ab}_M = \delta^{ab}\partial_M - g_m f^{abc}\mathcal{A}^c_M$. From (IV.21) the components of the curvature are transformed in the adjoint representation $\delta\mathcal{F}^a_{MN} = g_m f^{abc}\mathcal{F}^b_{MN}\alpha^c(x,y)$.

The Hamiltonian description of the theory goes along the same lines of Sect. II A. The conjugate momentum to $\mathcal{A}^a_M$ is denoted by π^M_a. The canonical analysis yields the following first–class constraints:

$$\phi^{(1)}_a = \pi^0_a(x,y) \approx 0 \tag{IV.22a}$$

$$\phi^{(2)}_a = \mathcal{D}^{ab}_I\pi^I_b(x,y) \approx 0 \tag{IV.22b}$$

where I labels all spatial components of $\mathcal{M}$. Therefore the number of physical degrees of freedom is $(N^2-1)m - 2(N^2-1) = (N^2-1)(m-2)$ per spatial point $(\mathbf{x}, y)$.

The corresponding gauge algebra has the structure of Eq. (II.11) with the corresponding coupling constant g_m

$$\{\phi_a^{(2)}[u], \phi_b^{(2)}[v]\}_{SU(N,\mathcal{M})} = g_m f_{abc}\, \phi_c^{(2)}[uv] \ . \tag{IV.23}$$

where the Poisson bracket $\{\cdot,\cdot\}_{SU(N,\mathcal{M})}$ is calculated in terms of canonical conjugate pairs $(\mathcal{A}_M^a, \pi_a^M)$. In the same fashion, gauge transformations (IV.21) can be obtained via the gauge generator (II.12), where $\alpha^a = \alpha^a(x,y)$, by using

$$\delta A_M^a = \{\mathcal{A}_M^a, G\}_{SU(N,\mathcal{M})} \ . \tag{IV.24}$$

We now perform the transition from the $SU(N,\mathcal{M})$ variables to the natural variables that arise in the effective theory after compactification.

C. Compactified theory and the $SU(N,\mathcal{M}^4)$ description

For the sake of simplicity, from now on we focus on the case $n = 1$, that is, five–dimensional $SU(N,\mathcal{M})$ Yang-Mills theory; however, most of our results can be extended to an arbitrary n. In five dimensions, the theory consists of $3(N^2-1)$ true degrees of freedom per spatial point $(\mathbf{x}, y)$

The components of the connection $\mathcal{A}_M^a$ find a natural split into $\mathcal{A}_\mu^a(x,y)$ and $\mathcal{A}_5^a(x,y)$ and following Ref. [6], we assume a compact extra dimension homotopically equivalent to the circle S^1 of radius R. The $\mathcal{A}_\mu^a(x,y)$ and $\mathcal{A}_5^a(x,y)$ fields are assumed to be periodic with respect the fifth coordinate, so they can be expressed as Fourier series. Since, in general, not all the zero modes in the Fourier series have associated a standard counterpart, it is desirable to eliminate some of these degrees of freedom by imposing extra symmetries acting on the fifth coordinate. One possibility is to demand that the fields of the theory obey some definite parity property under the reflection $y \to -y$. Under this assumption, Fourier series would involve only even or only odd Fourier expansions. To implement this symmetry, one replaces S^1 by S^1/Z_2 in which y is identified with $-y$. Following Ref. [6], we assume that $\mathcal{A}_\mu^a(x,y)$ and $\mathcal{A}_5^a(x,y)$ are, respectively, even and odd under the reflection $y \to -y$. In five dimensions, these requirements imply that curvature components $\mathcal{F}_{\mu\nu}^a(x,y)$ and $\mathcal{F}_{\mu 5}^a(x,y)$ display even and odd parity in the extra dimension, respectively. Under these assumptions, the following Fourier expansions are allowed:

$$\mathcal{A}_\mu^a(x,y) = \frac{1}{\sqrt{R}}\mathcal{A}_\mu^{(0)a}(x) + \sqrt{\frac{2}{R}}\sum_{m=1}^{\infty}\mathcal{A}_\mu^{(m)a}(x)\cos\left(2\pi\frac{my}{R}\right) \ , \tag{IV.25a}$$

$$\mathcal{A}_5^a(x,y) = \sqrt{\frac{2}{R}}\sum_{m=1}^{\infty}\mathcal{A}_5^{(m)a}(x)\sin\left(2\pi\frac{my}{R}\right) \ , \tag{IV.25b}$$

$$\mathcal{F}_{\mu\nu}^a(x,y) = \frac{1}{\sqrt{R}}\mathcal{F}_{\mu\nu}^{(0)a}(x) + \sqrt{\frac{2}{R}}\sum_{m=1}^{\infty}\mathcal{F}_{\mu\nu}^{(m)a}(x)\cos\left(2\pi\frac{my}{R}\right) \ , \tag{IV.25c}$$

$$\mathcal{F}_{\mu 5}^a(x,y) = \sqrt{\frac{2}{R}}\sum_{m=1}^{\infty}\mathcal{F}_{\mu 5}^{(m)a}(x)\sin\left(2\pi\frac{my}{R}\right) \ . \tag{IV.25d}$$

In particular, it will be important to make the analogy between Eqs. (IV.25a) and (IV.25b) and the point transformations (II.14).

Following the compactification scheme introduced in [6] one obtains the Fourier components of the curvature in terms of the gauge–field components

$$\mathcal{F}_{\mu\nu}^{(0)a} = F_{\mu\nu}^{(0)a} + g f^{abc} A_\mu^{(m)b} A_\nu^{(m)c} \ , \tag{IV.26a}$$

$$\mathcal{F}_{\mu\nu}^{(m)a} = \mathcal{D}_\mu^{(0)ab} A_\nu^{(m)b} - \mathcal{D}_\nu^{(0)ab} A_\mu^{(m)b} + g f^{abc}\Delta_{mrn} A_\mu^{(r)b} A_\nu^{(n)c} \ , \tag{IV.26b}$$

$$\mathcal{F}_{\mu 5}^{(m)a} = \mathcal{D}_\mu^{(0)ab} A_5^{(m)b} + \frac{2\pi m}{R} A_\mu^{(m)a} + g f^{abc}\Delta'_{mnr} A_\mu^{(r)b} A_5^{(n)c} \ , \tag{IV.26c}$$

where $\mathcal{D}_\mu^{(0)ab} = \delta^{ab}\partial_\mu - g f^{abc} A_\mu^{(0)c}$, the coupling constant $g = g_5/\sqrt{R}$, and

$$F_{\mu\nu}^{(0)a} = \partial_\mu A_\nu^{(0)a} - \partial_\nu A_\mu^{(0)a} + g f^{abc} A_\mu^{(0)b} A_\nu^{(0)c} \ . \tag{IV.27}$$

In addition

$$\Delta_{mrn} = \frac{1}{\sqrt{2}}\left(\delta_{r,m+n} + \delta_{m,r+n} + \delta_{n,r+m}\right) \ , \tag{IV.28a}$$

$$\Delta'_{mrn} = \frac{1}{\sqrt{2}}\left(\delta_{m,r+n} + \delta_{r,m+n} - \delta_{n,r+m}\right) \ . \tag{IV.28b}$$

Notice that there is a clear resemblance between Eqs. (II.16) and (IV.26). In the same fashion as the $su(3)$-valued curvature in our toy model was decomposed into well defined objects ($F^{\bar{a}}_{\mu\nu}$, $Y_{\mu\nu}$, and $F^{8}_{\mu\nu}$) under the $SU(2)$ subgroup by means of the point transformation (II.14), we will show that Eqs. (IV.26) represent the decomposition of the pure $SU(N,\mathcal{M})$ Yang-Mills curvature into well defined objects ($\mathcal{F}^{(0)a}_{\mu\nu}$, $\mathcal{F}^{(m)a}_{\mu\nu}$, and $\mathcal{F}^{(0)a}_{\mu 5}$) under the subgroup $SU(N,\mathcal{M}^4)$ (see Eq. (IV.36)). Also, this decomposition represents a map from covariant objects under $ISO(1,4)$ Poincaré group into covariant objects under the standard $ISO(1,3)$ one.

Integrating out the extra dimension after Fourier expanding (IV.20) yields the following effective Lagrangian, *cf.* (II.17):

$$\mathcal{L}_{SU(N,\mathcal{M}^4)} = -\frac{1}{4}\left(\mathcal{F}^{(0)a}_{\mu\nu}\mathcal{F}^{(0)a\mu\nu} + \mathcal{F}^{(m)a}_{\mu\nu}\mathcal{F}^{(m)a\mu\nu} + 2\,\mathcal{F}^{(m)a}_{\mu 5}\mathcal{F}^{(m)a\mu 5}\right) \ . \tag{IV.29}$$

The analysis of the toy model in Sec. II suggests that Fourier expansions of gauge fields, (IV.25a) and (IV.25b), can be treated as a point transformation which connects the natural coordinates in the pure $SU(N,\mathcal{M})$ five–dimensional Yang-Mills theory ($\mathcal{A}^a_M$) and the built-in coordinates ($\mathcal{A}^{(0)a}_{\mu}$, $\mathcal{A}^{(m)a}_{\mu}$, and $\mathcal{A}^{(m)a}_{5}$) of the effective Lagrangian (IV.29). In this framework, gauge transformations (IV.21) are mapped by (IV.25a) and (IV.25b) into

$$\delta A^{(0)a}_{\mu} = \mathcal{D}^{(0)ab}_{\mu}\alpha^{(0)b} + gf^{abc}A^{(m)b}_{\mu}\alpha^{(m)c} \ , \tag{IV.30a}$$
$$\delta A^{(m)a}_{\mu} = gf^{abc}A^{(m)b}_{\mu}\alpha^{(0)c} + \mathcal{D}^{(mn)ab}_{\mu}\alpha^{(n)b} \ , \tag{IV.30b}$$
$$\delta A^{(m)a}_{5} = gf^{abc}A^{(m)b}_{5}\alpha^{(0)c} + \mathcal{D}^{(mn)ab}_{5}\alpha^{(n)b} \ , \tag{IV.30c}$$

after the extra dimension is integrated out. The parameters $\alpha^{(0)a}(x)$ and $\alpha^{(m)a}(x)$ are the Fourier components in the expansion of $\alpha^a(x,y) = \alpha^a(x,-y)$. In expressions (IV.30) the following quantities have been defined:

$$\mathcal{D}^{(mn)ab}_{\mu} = \delta^{mn}\mathcal{D}^{(0)ab}_{\mu} - gf^{abc}\Delta_{mrn}A^{(r)c}_{\mu} \ , \tag{IV.31a}$$
$$\mathcal{D}^{(mn)ab}_{5} = -\frac{2\pi m}{R}\delta^{mn}\delta^{ab} - gf^{abc}\Delta'_{mrn}A^{(r)c}_{5} \ . \tag{IV.31b}$$

In analogy with Eqs. (II.19) and (II.20), the SGT and NSGT are defined for pure Yang-Mills theory. The SGT are defined from (IV.30) by restricting the gauge parameters of $SU(N,\mathcal{M})$, $\alpha^a(x,y)$, to the standard $\mathcal{M}^4$ manifold, that is

$$\delta_{\rm s} A^{(0)a}_{\mu} = \mathcal{D}^{(0)ab}_{\mu}\alpha^{(0)b}, \tag{IV.32a}$$
$$\delta_{\rm s} A^{(m)a}_{\mu} = gf^{abc}A^{(m)b}_{\mu}\alpha^{(0)c} \ , \tag{IV.32b}$$
$$\delta_{\rm s} A^{(m)a}_{5} = gf^{abc}A^{(m)b}_{5}\alpha^{(0)c} \ . \tag{IV.32c}$$

In analogy with the gauge fields $W^{\bar{a}}_{\mu}$ under $SU(2)$ (II.19a), the Fourier component $A^{(0)a}_{\mu}$ becomes a gauge field with respect to $SU(N,\mathcal{M}^4)$. The matter field Y_μ, a doublet that is transformed in the fundamental representation (II.19b), is comparable with the excited KK modes $A^{(n)a}_{\mu}$ which transform in the adjoint representation of $SU(N,\mathcal{M}^4)$. In addition, $A^{(n)5}_{\mu}$ transform as matter fields in the adjoint representation of $SU(N,\mathcal{M}^4)$. In contrast, the NSGT for pure Yang-Mills theory are obtained from (IV.30) by setting $\alpha^{(0)a} \equiv 0$, that is (*cf.* (II.20))

$$\delta_{\rm ns} A^{(0)a}_{\mu} = gf^{abc}A^{(m)b}_{\mu}\alpha^{(m)c} \ , \tag{IV.33a}$$
$$\delta_{\rm ns} A^{(m)a}_{\mu} = \mathcal{D}^{(mn)ab}_{\mu}\alpha^{(n)b} \ , \tag{IV.33b}$$
$$\delta_{\rm ns} A^{(m)a}_{5} = \mathcal{D}^{(mn)ab}_{5}\alpha^{(n)b} \ . \tag{IV.33c}$$

Gauge invariance of (IV.29) under (IV.30) is guaranteed since the latter imply the following variations at the level of the Fourier components of the curvature:

$$\delta\mathcal{F}^{(0)a}_{\mu\nu} = gf^{abc}\left(\mathcal{F}^{(0)b}_{\mu\nu}\alpha^{(0)c} + \mathcal{F}^{(m)b}_{\mu\nu}\alpha^{(m)c}\right) \ , \tag{IV.34a}$$
$$\delta\mathcal{F}^{(m)a}_{\mu\nu} = gf^{abc}\left(\mathcal{F}^{(m)b}_{\mu\nu}\alpha^{(0)c} + \left(\delta_{mn}\mathcal{F}^{(0)b}_{\mu\nu} + \Delta_{mrn}\mathcal{F}^{(r)b}_{\mu\nu}\right)\alpha^{(n)c}\right) \ , \tag{IV.34b}$$
$$\delta\mathcal{F}^{(m)a}_{\mu 5} = gf^{abc}\left(\mathcal{F}^{(m)b}_{\mu 5}\alpha^{(0)c} + \Delta'_{mrn}\mathcal{F}^{(r)b}_{\mu 5}\alpha^{(n)c}\right) \ . \tag{IV.34c}$$

It is not difficult to see that the effective Lagrangian $\mathcal{L}_{SU(N,\,\mathcal{M}^4)}$ is invariant under these transformations. Therefore, Eqs. (IV.30) are genuine gauge transformations of the effective theory.

It is worth noticing that the scalar fields $A_5^{(m)a}$ can be eliminated altogether via a particular NSGT. Consider a NSGT with infinitesimal gauge parameters given by $\alpha^{(m)a}(x) = (R/m)A_5^{(m)a}$ [6]. Then, from Eq. (IV.33c), we can see that $A_5^{(m)a} \to A_5^{\prime(m)a} = 0$. This result shows that the $A_5^{(m)a}(x)$ scalar fields are in fact pseudo Goldstone bosons.

It is important to stress that the invariance of the effective theory (IV.29) under the transformations (IV.30) is by no means immediate. A direct calculation of the curvature variations (IV.34) from (IV.30) give raise to the following extra terms quadratic in g:

$$\begin{aligned}\Delta_{\mu\nu}^{(m)a} =& - g^2 \left[f_{abc}f_{bde}(\delta_{pq}\delta_{mn} + \Delta_{rpq}\Delta_{rmn}) + f_{adb}f_{bce}(\delta_{nq}\delta_{mp} + \Delta_{rnq}\Delta_{rmp}) \right.\\ & \left. + f_{abe}f_{bcd}(\delta_{np}\delta_{mq} + \Delta_{rnp}\Delta_{rmq}) \right] A_\mu^{(p)d} A_\nu^{(q)e} \alpha^{(n)c} \qquad \text{(IV.35a)}\\ \widetilde{\Delta}_{\mu 5}^{(m)a} =& - g^2 \left[f_{abc}f_{bde}\Delta'_{rqp}\Delta'_{rmn} + f_{adb}f_{bce}\Delta'_{rqn}\Delta'_{rmp} \right.\\ & \left. + f_{abe}f_{bcd}(\delta_{np}\delta_{mq} + \Delta_{npr}\Delta'_{mqr}) \right] A_\mu^{(p)d} A_5^{(q)e} \alpha^{(n)c} \qquad \text{(IV.35b)}\end{aligned}$$

in Eqs. (IV.34b) and (IV.34c), respectively. These terms, that would destroy the invariance of the effective Lagrangian $\mathcal{L}_{SU(N,\mathcal{M}^4)}$ under (IV.30), are necessarily zero by consistency with the Fourier transformation (IV.25). The variation of curvatures $\delta\mathcal{F}_{MN}^a = g_5 f_{abc}\mathcal{F}_{MN}^b\alpha^c(x,y)$ is duly mapped into Eqs. (IV.34) under the point transformation (IV.25). We will discuss further this point within the Hamiltonian formalism of the theory.

The SGT (IV.32) induce the corresponding SGT at the curvature level. From Eqs. (IV.34), all Fourier components of $\mathcal{F}_{MN}^a$ do covariantly transform under the symmetry group of SGT, $SU(N, \mathcal{M}^4)$,

$$\delta_{\rm s}\mathcal{F}_{\mu\nu}^{(0)a} = g f^{abc}\mathcal{F}_{\mu\nu}^{(0)b}\alpha^{(0)c} \ , \qquad \text{(IV.36a)}$$

$$\delta_{\rm s}\mathcal{F}_{\mu\nu}^{(m)a} = g f^{abc}\mathcal{F}_{\mu\nu}^{(m)b}\alpha^{(0)c} \ , \qquad \text{(IV.36b)}$$

$$\delta_{\rm s}\mathcal{F}_{\mu 5}^{(m)a} = g f^{abc}\mathcal{F}_{\mu 5}^{(m)b}\alpha^{(0)c} \ . \qquad \text{(IV.36c)}$$

The phase space description of this theory allows us to define the gauge generators associated to the so–called SGT and NSGT defined above. The canonical analysis of the effective $SU(N, \mathcal{M}^4)$ Lagrangian (IV.29) goes along the same line as Sect. B2 of Ref. [6]. The conjugate momenta are given by

$$\pi_a^{(0)\mu} = \mathcal{F}_a^{(0)\mu 0} \ , \qquad \text{(IV.37a)}$$

$$\pi_a^{(n)\mu} = \mathcal{F}_a^{(n)\mu 0} \ , \qquad \text{(IV.37b)}$$

$$\pi_a^{(0)5} = \mathcal{F}_a^{(n)50} \ . \qquad \text{(IV.37c)}$$

It is worth noticing, from Eqs. (IV.32) and (IV.36), that canonical pairs are well defined objects with respect to $SU(N, \mathcal{M}^4)$. In addition, the Fourier expansions (IV.25c) and (IV.25d) together with $\pi_a^M = \mathcal{F}_a^{M0}$ allow us to write

$$\pi_a^\mu(x,y) = \frac{1}{\sqrt{R}}\pi_a^{(0)\mu}(x) + \sqrt{\frac{2}{R}}\sum_{m=1}^{\infty}\pi_a^{(m)\mu}(x)\cos\left(2\pi\frac{my}{R}\right) \ , \qquad \text{(IV.38a)}$$

$$\pi_a^5(x,y) = \sqrt{\frac{2}{R}}\sum_{m=1}^{\infty}\pi_a^{(m)5}(x)\sin\left(2\pi\frac{my}{R}\right) \ . \qquad \text{(IV.38b)}$$

These expressions relate the conjugate momenta inherent in the pure $SU(N, \mathcal{M})$ Yang-Mills theory and those presented in the effective $SU(N, \mathcal{M}^4)$ theory. Moreover, they are equivalent to (II.29).

The temporal component of (IV.37a) and (IV.37b) define the following primary constraints:

$$\phi_a^{(1)(0)} = \pi_a^{(0)0} \approx 0 \ , \qquad \text{(IV.39a)}$$

$$\phi_a^{(1)(n)} = \pi_a^{(n)0} \approx 0 \ . \qquad \text{(IV.39b)}$$

The primary Hamiltonian takes the form (*cf.* (II.32))

$$\mathcal{H}_{SU(N,\,\mathcal{M}^4)}^{(1)} = \mathcal{H}_{SU(N,\,\mathcal{M}^4)} + \mu^{(0)a}\phi_a^{(1)(0)} + \mu^{(n)a}\phi_a^{(1)(n)} \ , \qquad \text{(IV.40)}$$

where besides the linear combination of primary constraints, with Lagrange multipliers $\mu^{(0)a}$ and $\mu^{(n)a}$ as coefficients, the canonical Hamiltonian is (*cf.* (II.33))

$$\mathcal{H}_{SU(N,\mathcal{M}^4)} = \frac{1}{2}\left(\pi_a^{(0)i}\pi_a^{(0)i} + \pi_a^{(n)i}\pi_a^{(n)i} + \pi_a^{(n)5}\pi_a^{(n)5}\right) + \frac{1}{4}\left(\mathcal{F}_a^{(0)ij}\mathcal{F}_{ij}^{(0)a} + 2\mathcal{F}_a^{(n)i5}\mathcal{F}_{i5}^{(n)a}\right) - A_0^{(0)a}\phi_a^{(2)(0)} - A_0^{(n)a}\phi_a^{(2)(n)} \ , \tag{IV.41}$$

where $\phi_a^{(2)(0)}$ and $\phi_a^{(2)(n)}$ are functions of phase space that will be specified after presenting a couple of key results useful for the rest of the discussion.

Proposition IV.1 *The Fourier expansion of gauge fields and conjugate momenta, Eqs.* (IV.25a), (IV.25b) *and* (IV.38)*, define a canonical transformation.*

The proof of this proposition is collected in the Appendix A. This proposition ensures that $\{\cdot,\cdot\}_{SU(N,\mathcal{M})} = \{\cdot,\cdot\}_{SU(N,\mathcal{M}^4)}$, where $\{\cdot,\cdot\}_{SU(N,\mathcal{M}^4)}$ indicates the Poisson bracket with respect to $(A_\mu^{(0)a}, \pi_a^{(0)\mu})$, $(A_\mu^{(n)a}, \pi_a^{(n)\mu})$, and $(A_5^{(n)a}, \pi_a^{(n)5})$. Because there exists a spacetime independent canonical transformation between the pure $SU(N,\mathcal{M})$ Yang-Mills theory and the effective theory $SU(N,\mathcal{M}^4)$, it immediately follows that both canonical Hamiltonians $\mathcal{H}_{SU(N,\mathcal{M})}$ and $\mathcal{H}_{SU(N,\mathcal{M}^4)}$ are mapped into each other via such a transformation, as can be proved by direct calculation. However, in a singular theory the time evolution is governed by the primary Hamiltonian and not by the canonical one. An important observation is the following. If in a general singular theory of fields there is a spacetime independent canonical transformation which connects two primary Hamiltonians corresponding to two different formulations of the same theory, that is, if such transformation maps one set of primary constraints into the other one, thus both formulations must have the same number of generations of constraints (tertiary, quartic, etc.). This is an immediate consequence of the relation between the Poisson brackets in the two different formulations. Another consequence is that the set of secondary (tertiary, quartic, etc.) constraints in one of the formulation is necessarily mapped into the corresponding set of constraints in the other formulation via the canonical transformation. Hence it is useful the following result

Proposition IV.2 *The set of primary constraints* (IV.22a) *of the five dimensional pure $SU(N,\mathcal{M})$ Yang-Mills theory is faithfully mapped into the set of primary constraints* (IV.39) *of the $SU(N,\mathcal{M}^4)$ Yang-Mills theory.*

The proof of this proposition is straightforward from Eq. (IV.38a) and the linear independence of trigonometric functions. Moreover, it can be extended to the case of m dimensional pure $SU(N,\mathcal{M})$ Yang-Mills theory and its compactification down to four dimensions.

Propositions IV.1 and IV.2 ensure that secondary constraints

$$\phi_a^{(2)(0)} = \mathcal{D}_i^{(0)ab}\pi_b^{(0)i} - gf^{abc}\left(A_i^{(n)c}\pi_b^{(n)i} + A_5^{(m)c}\pi_b^{(m)5}\right) \approx 0 \ , \tag{IV.42a}$$

$$\phi_a^{(2)(n)} = \mathcal{D}_i^{(nm)ab}\pi_b^{(m)i} - \mathcal{D}_5^{(nm)ab}\pi_b^{(m)5} - gf^{abc}A_i^{(n)c}\pi_b^{(0)i} \approx 0 \ , \tag{IV.42b}$$

that emerge in the canonical Hamiltonian (IV.41), can be also calculated from (IV.22b) via the transformations (IV.25a), (IV.25b) and (IV.38). Indeed, the result of that calculation matches with Eqs. (IV.42). Less–trivial outcomes of the considerations above are the following. Firstly, the effective theory must not present either tertiary or higher constraint generations. Secondly, the gauge algebra of the effective theory can be obtained via the canonical transformation from the gauge algebra (IV.23) of the pure five dimensional $SU(N,\mathcal{M})$ Yang-Mills theory. In fact,

$$\{\phi_a^{(2)(0)}[u], \phi_b^{(2)(0)}[v]\} = gf_{abc}\phi_c^{(2)(0)}[uv] \ , \tag{IV.43a}$$

$$\{\phi_a^{(2)(0)}[u], \phi_b^{(2)(n)}[v]\} = gf_{abc}\phi_c^{(2)(n)}[uv] \ , \tag{IV.43b}$$

$$\{\phi_a^{(2)(m)}[u], \phi_b^{(2)(n)}[v]\} = gf_{abc}\left(\delta_{mn}\phi_c^{(2)(0)}[uv] + \Delta_{mnr}\phi_c^{(2)(r)}[uv]\right) \ . \tag{IV.43c}$$

which coincides with Eqs. (68)-(70) of Ref. [6].

The gauge generator that reproduces the gauge transformations (IV.30) is the sum of the SGT ($G_{\rm s}$) plus the NSGT ($G_{\rm ns}$) generators, where

$$G_{\rm s} = \left(\mathcal{D}_0^{(0)ab}\alpha^{(0)b}\right)\phi_a^{(1)(0)} + gf_{abc}A_0^{(n)b}\alpha^{(0)c}\phi_a^{(1)(n)} - \alpha^{(0)a}\phi_a^{(2)(0)} \tag{IV.44a}$$

$$G_{\rm ns} = gf_{abc}A_0^{(n)b}\alpha^{(m)c}\phi_a^{(1)(0)} + \left(\mathcal{D}^{(mn)ab}\alpha^{(n)b}\right)\phi_a^{(1)(n)} - \alpha^{(m)a}\phi_a^{(2)(n)} \tag{IV.44b}$$

The sum $G_{\rm s}+G_{\rm ns}$ is the image under the canonical transformation mentioned in Prop. IV.1 of the gauge generator that reproduces gauge transformations (IV.21) in the five dimensional case.

In this paper we take the point of view that if one can find a complete set of gauge transformations at the Hamiltonian level, one automatically gets a complete set of gauge transformations at the Lagrangian level [25]. This implies that there are no more gauge transformations of the Lagrangian than those generated by (IV.44), which in turn correspond to (IV.30). Therefore, the effective Lagrangian (IV.29) must be invariant under these transformations, so that any extra term in the calculation of $\delta\mathcal{L}_{SU(N,\mathcal{M}^4)}$ must be either identically zero or a surface term. In this regard we argue that the extra terms (IV.35a) and (IV.35b) must vanish since they do not include any derivative, hence they cannot be rewritten as a surface term.

We end this section with a heuristic counting of true degrees of freedom in the effective theory. Let us take for the moment "truncated Fourier expansions" up to some order K, so that, letting $K\to\infty$ will precisely yield $(\mathcal{A}^a_\mu(x,y),\pi^\mu_a(x,y))$ and $(\mathcal{A}^a_5(x,y),\pi^5_a(x,y))$ in terms of $(A^{(0)a}_\mu(x),\pi^{(0)\mu}_a(x))$, $(A^{(n)a}_\mu(x),\pi^{(n)\mu}_a(x))$, $(A^{(n)a}_5(x),\pi^{(n)5}_a(x))$ and trigonometric functions. In other words, K quantifies the contribution from the extra dimension in the "truncated Fourier expansions". The number of canonical pairs and first–class constraints in the truncated version are $2\times[4(N^2-1)+4K(N^2-1)+K(N^2-1)]$ and $2(N^2-1)+2K(N^2-1)$, respectively. Thus, the number of true degrees of freedom when K is large but finite is $N_0(K)=2(N^2-1)+3K(N^2-1)$ per spatial point $(\mathbf{x})$. Allowing $K\to\infty$ this number of true degrees of freedom becomes infinity because one is also counting the continuum contribution of the extra dimension. In order to obtain the number of true degrees of freedom per spatial point $(\mathbf{x},y)$ one needs to take the ratio N_0/K before considering $K\to\infty$. After this process is done we have that the number of true degrees of freedom per $(\mathbf{x},y)$ point is $3(N^2-1)$, which coincides with the corresponding number in the pure $SU(N,\mathcal{M})$ Yang-Mills theory.

V. FINAL REMARKS

In this paper, we have presented a comprehensive study of the gauge structure of Yang-Mills theories with compactified extra dimensions. As it was shown in Ref. [6], the gauge structure of the effective theory that emerges after compactification is subtle because it involves an infinite number of gauge parameters. Moreover, the switch from the original to the effective theory via compactification is highly nontrivial, this is due to the fact that at first glance it is not clear if both theories are indeed equivalent. We have clarified these points by appealing to the fundamental concept of canonical transformation and the notion of hidden symmetry.

Although the idea of hidden symmetry is well known in the context of theories with SSB, we have extended this notion to comprise more general scenarios. The main idea behind a hidden symmetry as we have presented it, is the following. Consider a gauge system governed by a group G, denote by q^a the coordinates that transform as components of a $g-$valued connection and let p_a be their canonical conjugate momenta; that is, $\{q^a,p_b\}=\delta^a_b$, with $a,b,\cdots$ collectively denoting discrete and continuous indices. Assume that the action $S[q]$ of the theory does not involve tensorial representations of G, the theory comes in terms of q^a only. Let H be a nontrivial subgroup of G, say, generated by a subset $\{T^{\bar a}\}$ of the generators of G. From the H perspective not all q's transform as connection components, but only $q^{\bar a}$; the remainder q's arise in a tensorial representation of H. We have considered the canonical transformation $(q^a,p_a)\mapsto(Q^{\bar a},Q^{\hat a},P_{\bar a},P_{\hat a})$, where $q^{\bar a}\equiv Q^{\bar a}$ and $P_{\bar a}\equiv p_{\bar a}$, the conjugate pairs $(Q^{\hat a},P_{\hat a})$ stand for those coordinates that furnish a tensorial representation of H. At the configuration–space level $(q^a)\mapsto(Q^{\bar a},Q^{\hat a})$ is given by a point a transformation. The observation we have stressed is that although from the H perspective we have a theory with $h-$valued connection components $q^{\bar a}$ and coordinates $Q^{\hat a}$ that transform in tensorial representations of H, which could be interpreted as matter fields, both sets of coordinates can be actually seen as components of a single $g-$valued connection from the G perspective. Is in this sense that the larger symmetry G is not lost, but hidden in the theory defined by $S[q^{\bar a},Q^{\hat a}]$.

These considerations were illustrated in detail in sections II and III by using $G=SU(3)$ and $H=SU(2)$. In theories where SSB is present via a mechanism where the gauge group G is broken into a subgroup H, using the H perspective, as explained above, is practical as it emphasizes that one has to take into account the group G underlying the theory. This is particularly useful in the quantization of such theories. The corresponding NSGT in pure Yang-Mills theory with one UED were similarly implemented to fix a unitary gauge.

Pure Yang-Mills theory with one UED also lies in our context of hidden symmetry. These theories are formulated to be invariant under a gauge group $SU(N,\mathcal{M})$, with $\mathcal{M}^m=\mathcal{M}^4\times\mathcal{N}^n$ the spacetime manifold, and under the corresponding Poincaré group $ISO(1,m-1)$. Then, it is assumed that the $\mathcal{N}^n$ manifold is compact and that the fundamental objects of the theory, namely, the gauge fields, $\mathcal{A}^a_M(x,y)$, and the gauge parameters, $\alpha^a(x,y)$, are periodic functions with respect to the compact coordinates y. To recover the theory governed by the groups $SU(N,\mathcal{M}^4)$ and $ISO(1,3)$, some parity requirements with respect to the extra coordinates are imposed on the fields and gauge parameters. We established a map from the original objects $\mathcal{A}^a_M(x,y)$ and $\alpha^a(x,y)$ into new objects $(A^{(0,\cdots)a}_\mu(x)\ ,A^{(m,\cdots)a}_\mu(x),\ \cdots,A^{(m,\cdots)a}_{\bar\mu}(x),\ \cdots)$ and $(\alpha^{(0,\cdots)a}(x),\ \alpha^{(m,\cdots)a}(x),\cdots)$ via a Fourier transformation; the

latter sets being the Fourier modes. The periodicity and parity properties of the conjugate momenta $\pi^a_\mu(x,y)$ and $\pi^a_{\bar\mu}(x,y)$ are inherit from the fields $\mathcal{A}^a_\mu(x,y)$ and $\mathcal{A}^a_{\bar\mu}(x,y)$, so that corresponding Fourier series can be established. These maps were shown to be a canonical transformation that allows to establish the correspondence between the $SU(N,\mathcal{M})$ and $SU(N,\mathcal{M}^4)$ perspectives. Let us to examine more closely these Fourier transformations.

- **Covariance.** The maps

$$\mathcal{A}^a_\mu(x,y) \;\mapsto\; \left(A^{(0,\cdots)a}_\mu(c),\; A^{(m,\cdots)a}_\mu(x),\; \cdots\right), \tag{V.1}$$

$$\mathcal{A}^a_{\bar\mu}(x,y) \;\mapsto\; \left(A^{(m,\cdots)a}_{\bar\mu}(x),\; \cdots\right), \tag{V.2}$$

$$\alpha^a(x,y) \;\mapsto\; \left(\alpha^{(0,\cdots)a}(x),\; \alpha^{(m,\cdots)a}(x),\; \cdots\right), \tag{V.3}$$

  are all transformations from covariant objects of $SU(N,\mathcal{M})$ and $ISO(1,m-1)$ to covariant objects of the standard groups $SU(N,\mathcal{M}^4)$ and $ISO(1,3)$. We have shown that the maps

$$\left(\mathcal{A}^a_\mu(x,y),\; \pi^a_\mu(x,y)\right) \;\mapsto\; \left(A^{(0,\,\cdots)a}_\mu(x),\; \pi^{(0,\,\cdots)a}_\mu\right),\; \left(A^{(m,\,\cdots)a}_\mu(x),\; \pi^{(m,\,\cdots)a}_\mu\right), \cdots, \tag{V.4}$$

$$\left(\mathcal{A}^a_{\bar\mu}(x,y),\; \pi^a_{\bar\mu}(x,y)\right) \;\mapsto\; \left(A^{(m,\,\cdots)a}_{\bar\mu}(x),\; \pi^{(m,\,\cdots)a}_{\bar\mu}\right), \cdots, \tag{V.5}$$

  form a canonical transformation.

- **There is no spontaneous symmetry breaking.** As Lie groups, $SU(N,\mathcal{M})$ and $SU(N,\mathcal{M}^4)$ share the same number of generators, so the map from one to the other cannot involve a SSB. However, as gauge groups $SU(N,\mathcal{M})$ and $SU(N,\mathcal{M}^4)$ are quite different because their parameters are defined on different manifolds. Indeed the restriction of the $\alpha^a(x,y)$ parameters to the usual spacetime manifold $\mathcal{M}^4$ leads to the standard gauge group $SU(N,\mathcal{M}^4)$, so it is a subgroup of $SU(N,\mathcal{M})$ in this sense. Since there are no broken generators of the gauge group, the Higgs mechanism does not operate in the conventional sense in this case, as massless Goldstone bosons cannot emerge from the gauge group. The *pseudo Goldstone bosons* needed for the Higgs mechanism are provided by an explicit breaking of the Poincaré group $ISO(1,m-1)$ into the standard one $ISO(1,3)$. From these considerations, it is now clear that the map from $SU(N,\mathcal{M})$ to $SU(N,\mathcal{M}^4)$ corresponds to a more general scenario than the one considered in Sec. II for the groups $SU(3)$ and $SU(2)$. However, the essential ingredient of mapping covariant objects onto covariant objects is common to both scenarios.

- **SGT and NSGT.** In a map between conventional gauge groups, as that from $SU(3)$ to $SU(2)$, the SGT are associated to the generators of the subgroup H, whereas the NSGT are linked to the generators of G which do not belong to H. We have shown that the transition from the G to the H description is done via a canonical transformation. From the H perspective, the G symmetry is not manifest, but it is hidden. When the spontaneous breakdown of G into H is considered, one can use an specific NSGT to remove the pseudo Goldstone bosons from the theory. This gauge–fixing procedure defines the unitary gauge. When the map given by the Fourier series from $SU(N,\mathcal{M})$ into $SU(N,\mathcal{M}^4)$ is implemented, also SGT and NSGT [6] emerge. In this scenario, the SGT are the ones associated to the gauge group $SU(N,\mathcal{M}^4)$, whereas the NSGT have nothing to do with generators of a Lie group, but with certain type of transformations of the gauge fields $(A^{(0,\cdots)a}_\mu(c),\; A^{(m,\cdots)a}_\mu(x),\; \cdots)$ determined by the $(\alpha^{(m,\cdots)a}(x),\; \cdots)$ parameters. Although this is an important difference, we have shown that these transformations have an identical structure to those that arise in a map between conventional gauge groups. Moreover, we have shown that the pseudo–Goldstone bosons that arise from compactification can be removed of the theory using an specific NSGT (unitary gauge), just as it is done in conventional gauge theories. Indeed, through the paper we have found an close parallelism between both types of canonical maps.

- **Equivalence of the theories.** The fact that the map from $SU(N,\mathcal{M})$ into $SU(N,\mathcal{M}^4)$ is canonical becomes crucial to prove that the effective theory is completely equivalent to the original theory. It is not only necessary to show that they have the same number of degrees of freedom, but also that they share the same gauge algebra. Indeed, we have shown that the latter requirement implies the former. It is at this point that the canonical nature of the $SU(N,\mathcal{M}) \mapsto SU(N,\mathcal{M}^4)$ map plays a central role, as it is the invariance of the Poisson's bracket $\{\cdot,\cdot\}_{SU(N,\mathcal{M})} = \{\cdot,\cdot\}_{SU(N,\mathcal{M}^4)}$. This automatically implies that the canonical Hamiltonian is invariant, that is, $\mathcal{H}_{SU(N,\mathcal{M})} = \mathcal{H}_{SU(N,\mathcal{M}^4)}$, because the canonical transformation does not depend explicitly on the time. However, the evolution of the constraints in one or the other theory is not determined by the canonical Hamiltonian but by the primary one. We have proved that both theories share the same gauge algebra by showing that the gauge generator of one theory is mapped into the other.

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Appendix A: Fourier expansion as a canonical transformation

We will prove that Fourier expansion is a canonical transformation by showing that it maps conjugate canonical pairs into conjugate canonical pairs. We will make use of the following Poisson brackets among the gauge fields and their canonical conjugate momenta:

$$\begin{aligned}\{\mathcal{A}^a_M(x,y),\pi^N_b(x',y')\}_{SU(N,\mathcal{M})} &= \delta^a_b\delta^N_M\delta(x-x')\delta(y-y')\\ \{\mathcal{A}^a_M(x,y),\mathcal{A}^b_N(x',y')\}_{SU(N,\mathcal{M})} &= \{\pi^M_a(x,y),\pi^N_b(x',y')\}_{SU(N,\mathcal{M})}=0\,,\end{aligned}$$

as well as of the inverse Fourier transformations

$$\begin{aligned}\mathcal{A}^{(0)a}_\mu(x) &= \frac{1}{\sqrt{R}}\int \mathrm{d}y\,\mathcal{A}^a_\mu(x,y)\\ \mathcal{A}^{(m)a}_\mu(x) &= \sqrt{\frac{2}{R}}\int \mathrm{d}y\,\mathcal{A}^a_\mu(x,y)\cos\left(2\pi\frac{my}{R}\right)\\ \mathcal{A}^{(m)a}_5(x) &= \sqrt{\frac{2}{R}}\int \mathrm{d}y\,\mathcal{A}^a_5(x,y)\sin\left(2\pi\frac{my}{R}\right)\\ \pi^{(0)\mu}_a(x) &= \frac{1}{\sqrt{R}}\int \mathrm{d}y\,\pi^\mu_a(x,y)\\ \pi^{(m)\mu}_a(x) &= \sqrt{\frac{2}{R}}\int \mathrm{d}y\,\pi^\mu_a(x,y)\cos\left(2\pi\frac{my}{R}\right)\\ \pi^{(m)5}_a(x) &= \sqrt{\frac{2}{R}}\int \mathrm{d}y\,\pi^5_a(x,y)\sin\left(2\pi\frac{my}{R}\right)\,.\end{aligned}$$

In order to properly deal with the distribution characteristic of the Poisson brackets, we will use smooth smearing functions u and v defined on $\mathcal{M}^4$.

The nonvanishing Poisson brackets for the Fourier modes are the following: First, the zero components with four dimensional spacetime labels

$$\begin{aligned}&\{\mathcal{A}^{(0)a}_\mu[u],\pi^{(0)\nu}_b[v]\}_{SU(N,\mathcal{M}^4)}=\int \mathrm{d}^3x\,\mathrm{d}^3x'\,u(x)v(x')\{\mathcal{A}^{(0)a}_\mu(x),\pi^{(0)\nu}_b(x')\}_{SU(N,\mathcal{M}^4)}\\ &=\int \mathrm{d}^3x\,\mathrm{d}^3x'\mathrm{d}y\,\mathrm{d}y'\,u(x)v(x')\frac{1}{R}\{\mathcal{A}^a_\mu(x,y),\pi^\nu_b(x',y')\}_{SU(N,\mathcal{M})}\\ &=\int \mathrm{d}^3x\,\mathrm{d}^3x'\mathrm{d}y\,\mathrm{d}y'\,u(x)v(x')\frac{1}{R}\delta^a_b\delta^\nu_\mu\delta(x-x')\delta(y-y')=\delta^a_b\delta^\nu_\mu[uv]\,;\end{aligned}\tag{A.1}$$

second, the m modes with four dimensional spacetime labels

$$\begin{aligned}&\{\mathcal{A}^{(m)a}_\mu[u],\pi^{(n)\nu}_b[v]\}_{SU(N,\mathcal{M}^4)}=\int \mathrm{d}^3x\,\mathrm{d}^3x'\,u(x)v(x')\{\mathcal{A}^{(m)a}_\mu(x),\pi^{(n)\nu}_b(x')\}_{SU(N,\mathcal{M}^4)}\\ &=\int \mathrm{d}^3x\,\mathrm{d}^3x'\mathrm{d}y\,\mathrm{d}y'\,u(x)v(x')\frac{2}{R}\cos\left(2\pi\frac{my}{R}\right)\cos\left(2\pi\frac{ny'}{R}\right)\{\mathcal{A}^a_\mu(x,y),\pi^\nu_b(x',y')\}_{SU(N,\mathcal{M})}\\ &=\delta^a_b\delta^\nu_\mu\delta^{mn}[uv]\,;\end{aligned}\tag{A.2}$$

and finally, the m modes with the fifth component

$$\begin{aligned}&\{\mathcal{A}^{(m)a}_5[u],\pi^{(n)5}_b[v]\}_{SU(N,\mathcal{M}^4)}=\int \mathrm{d}^3x\,\mathrm{d}^3x'\,u(x)v(x')\{\mathcal{A}^{(m)a}_5(x),\pi^{(n)5}_b(x')\}_{SU(N,\mathcal{M}^4)}\\ &=\int \mathrm{d}^3x\,\mathrm{d}^3x'\mathrm{d}y\,\mathrm{d}y'\,u(x)v(x')\frac{2}{R}\sin\left(2\pi\frac{my}{R}\right)\sin\left(2\pi\frac{ny'}{R}\right)\{\mathcal{A}^a_5(x,y),\pi^5_b(x',y')\}_{SU(N,\mathcal{M})}\\ &=\delta^a_b\delta^{mn}[uv]\,.\end{aligned}\tag{A.3}$$

Thus, under the assumption that $(\mathcal{A}^a_M, \pi^M_a)$ are canonical pairs, one obtains that in pairs $(\mathcal{A}^{(0)a}_\mu, \pi^{(0)\mu}_a)$, $(\mathcal{A}^{(m)a}_\mu, \pi^{(m)\mu}_a)$ and $(\mathcal{A}^{(m)a}_5, \pi^{(m)5}_a)$ are canonical pairs.

Conversely, assuming that $(\mathcal{A}^{(0)a}_\mu, \pi^{(0)\mu}_a)$, $(\mathcal{A}^{(m)a}_\mu, \pi^{(m)\mu}_a)$ and $(\mathcal{A}^{(m)a}_5, \pi^{(m)5}_a)$ are canonical pairs, one obtains that $(\mathcal{A}^a_M, \pi^M_a)$ are canonical pairs. This is achieved using the Fourier transform and smear functions u and v defined in $\mathcal{M}$, therefore periodic in y. These functions will be asked to be even when calculating the Poisson brackets between $\mathcal{A}^a_\mu$ and π^ν_b , so that they can be expanded as follows

$$u(x,y) = \frac{1}{\sqrt{R}}\, u^{(0)}(x) + \sqrt{\frac{2}{R}} \sum_{m=1}^{\infty} u^{(m)}(x) \cos\left(2\pi \frac{my}{R}\right) \; ; \tag{A.4}$$

and we will demand they are odd when calculating the Poisson brackets between $\mathcal{A}^a_5$ and π^5_b, thus expanded as

$$u(x,y) = \sqrt{\frac{2}{R}} \sum_{m=1}^{\infty} u^{(m)}(x) \sin\left(2\pi \frac{my}{R}\right) \; . \tag{A.5}$$

In conclusion, from a set of conjugate pairs we obtain, via the Fourier transform, another set of conjugate pairs.

This proof can easily be extended in the presence of more extra dimensions, provided each of the extra dimensions is a compact manifold homotopically equivalent to S^1, and that the corresponding canonical pairs have suitable parity properties on the extra dimensions.

[1] The ATLAS Collaboration, *Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC*, Phys.Lett. **B716**, 1 (2012), arXiv:1207.7214 [hep-ex].

[2] The CMS Collaboration, *Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC*, arXiv:1207.7235 [hep-ex].

[3] F. Englert and R. Brout, Phys. Rev. Lett. **13**, 321 (1964); Phys. Rev. Lett. **13** 321E (1964); Peter W. Higgs, Phys. Rev. Lett. **13**, 508 (1964); G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble, Phys. Rev. Lett. **13**, 585 (1964); Phys. Rev. **155**, 1554 (1967).

[4] Y. Nambu, Phys. Rev. **117**, 648 (1960); J. Goldstone, Nuovo Cimento **19**, 154 (1961); J. Goldstone, A. Salam, S. Weinberg, Phys. Rev. **127**, 965 (1962).

[5] I. Antoniadis, Phys. Lett. B **246**, 377 (1990); N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, Phys. Lett. B **429**, 263 (1998); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, Phys. Lett. B **436**, 257 (1998).

[6] H. Novales-Sánchez and J. J. Toscano, Phys. Rev. D **82**, 116012 (2010).

[7] H. Novales-Sánchez and J. J. Toscano, Phys. Rev. D **84**, 05790 (2011); *ibid.* 076010 (2011).

[8] A. Cordero-Cid, M. Gómez-Bock, H. Novales-Sánchez, J. J. Toscano, arXiv:1108.2926 [hep-ph].

[9] K. R. Dienes, E. Dudas, and T. Gherghetta, Nucl. Phys. **B537**, 47 (1999); A. Mück, A. Pilaftsis, and R. Rückl, Phys. Rev. D **65**, 085037 (2002); N. Uekusa, Phys. Rev. D **75**, 064014 (2007); R. S. Chivukula, D. A. Dicus, and H. –J. He, Phys. Lett. B **525**, 175 (2002); S. De Curtis, D. Dominici, and J. R. Pelaez, Phys. Lett. B **554**, 164 (2003).

[10] G. Servant, and T. M. P. Tait, New J. Phys. **4**, 99 (2002); H. –C. Cheng, J. L. Feng, and K. T. Matchev, Phys. Rev. Lett. **89**, 211301 (2002); G. Servant, and T. M. P. Tait, Nucl. Phys. **B650**, 391 (2003); L. Bergström, T. Bringmann, M. Eriksson, and M. Gustafsson, JCAP **0504**, 004 (2005); S. Matsumoto, and M. Senami, Phys. Lett. B **633**, 671 (2006); M. Kakizaki, S. Matsumoto, Y. Sato, and M. Senami, Nucl. Phys. **B735**, 84 (2006); M. Kakizaki, S. Matsumoto, and M. Senami, Phys. Rev. D **74**, 023504 (2006); F. Burnell, and G. D. Kribs, Phys. Rev. D **73**, 015001 (2006); S. Matsumoto, J. Sato, M. Senami, and M. Yamanaka, Phys. Rev. D **76**, 043528 (2007); J. A. R. Cembranos, J. L. Feng, and L. E. Strigari, Phys. Rev. D **75**, 036004 (2007); M. Blennow, H. Melbeus, and T. Ohlsson, JCAP **1001**, 018 (2010).

[11] S. Matsumoto, J. Sato, M. Senami, and M. Yamanaka, Phys. Rev. D **76**, 043528 (2007); S. Matsumoto, J. Sato, M. Senami, and M. Yamanaka, Phys. Lett. B **647**, 466 (2007).

[12] F. J. Petriello, JHEP **0205** (2002) 003; T. Appelquist and H-U. Yee, Phys. Rev. D **67**, 055002 (2003); P. Bandyopadhyay, B. Bhattacherjee and A. Datta, JHEP **1003** (2010) 048.

[13] A. J. Buras, M. Spranger, and A. Weiler, Nucl. Phys. **B660**, 225 (2003); A. J. Buras, A. Poschenrieder, M. Spranger, and A. Weiler, Nucl. Phys. **B678**, 455 (2004); E. O. Iltan, JHEP **0402** (2004) 065; S. Khalil, and R. Mohapatra, Nucl. Phys. **B695**, 313 (2004).

[14] T. G. Rizzo, Phys. Rev. D **64**, 095010 (2001); H. –C. Cheng, K. T. Matchev, and M. Schmaltz, Phys. Rev. D **66**, 056006 (2002); C. Macesanu, C. D. McMullen, and S. Nandi, Phys. Rev. D **66**, 015009 (2002); G. Bhattacharyya, P. Dey, A. Kundu, A. Raychaudhuri, Phys. Lett. B **628**, 141 (2005); M. Battaglia, A. Datta, A. De Roeck, K. Kong, K. T. Matchev, JHEP **0507** (2005) 033; B. Bhattacherjee, and A. Kundu, Phys. Lett. B **627**, 137 (2005); B. Bhattacherjee, and A. Kundu, Phys. Lett. B **653**, 300 (2007); D. Hooper, and S. Profumo, Phys. Rept. **453**, 29 (2007); B. Bhattacherjee, A. Kundu, S. K. Rai, and S. Raychaudhuri, Phys. Rev. D **78**, 115005 (2008); P. Konar, K. Kong, K. T. Matchev, and M. Perelstein, New J. Phys. **11** 105004 (2009); S. Matsumoto, J. Sato, M. Senami, and M. Yamanaka, Phys. Rev. D **80**, 056006

(2009); G. Bhattacharyya, A. Datta, S. K. Majee, A. Raychaudhuri, Nucl. Phys. **B821**, 48 (2009); P. Bandyopadhyay, B. Bhattacherjee, and A. Datta, JHEP **1003**, 048 (2010); B. Bhattacherjee, A. Kundu, S. K. Rai, and S. Raychaudhuri, Phys. Rev. D **81**, 035021 (2010); D. Choudhury, A. Datta, and K. Ghosh, JHEP **1008**, 051 (2010); B. Bhattacherjee, and K. Ghosh, Phys. Rev. D **83**, 034003 (2011); A. Datta, A. Datta, and S. Poddar, Phys. Lett. B **712**, 219 (2012); A. Datta, U. K. Dey, A. Shaw, and A. Raychaudhuri, e–Print: arXiv:1205.4334 [hep–ph]; S. Chang, K. Y. Lee, S. Y. Shim, and J. Song, Phys. Rev. D **86**, 117503 (2012).

[15] A. Flores-Tlalpa, J. Montaño, H. Novales-Sánchez, F. Ramírez-Zavaleta, J. J. Toscano, Phys. Rev. D **83**, 016011 (2011).

[16] F. Pisano and V. Pleitez, Phys. Rev. D **46**, 410 (1992); P.H. Frampton, Phys. Rev. Lett. **69**, 2889 (1992).

[17] Dirac, P. A. M. *Lectures on Quantum Mechanics*. Belfer Graduate School of Sciences, Yeshiva University, (1964).

[18] Henneaux, M. and Teitelboim, C., *Quantization of Gauge Systems*. Princeton University Press (1992).

[19] L. Castellani, Ann. Phys **143**, 357 (1982).

[20] C. Becchi, A Rouet, and R. Stora, Commun. Math. Phys. 42, 127 (1975); Ann. Phys. (N.Y.) **98**, 287 (1976); I.V. Tyutin, FIAN (P.N: Lebedev Physical Institute of the USSR Academy of Science), Report No. 39, 1975.

[21] For a review of the BRST symmetry at both the classical and quantum levels within the context of the field-antifield formalism, see J. Gomis, J. Paris, and S. Samuel, Phys. Rep. 259, 1 (1995).

[22] J. Montaño, F. Ramírez-Zavaleta, G. Tavares-Velasco, and J. J. Toscano, Phys. Rev. D **72**, 055023 (2005); F. Ramírez-Zavaleta, G. Tavares-Velasco, and J. J. Toscano, Phys. Rev. D **75**, 075008 (2007).

[23] Goldstein, H. *Classical Mechanics*. Addison–Wesley Publishing Company, 2nd edn, (1980).

[24] J. M. Pons, Stud. Hist. Philos. Mod. Phys. **36**, 491 (2005).

[25] Henneaux, M., Teitelboim, C., and Zanelli, J. Nucl. Phys. **B332**, 169–188 (1990).